Lecture 2: A Segal model for \( \infty \)-modular operads
Yesterday:

\[ \text{ModOp} \xrightarrow{\approx} \text{Nerve} \overset{\approx}{\Rightarrow} \text{Set}^{\text{Segal}}_{\text{Segal}}. \]
A modular operad consists of a collection $P = \{P(n)\}$ where:

- Each $P(n)$ has a $\Sigma_n$-action;
- A family of equivariant associative compositions

$$P(n) \times P(m) \xrightarrow{o_{ij}} P(n + m - 2)$$
Yesterday:

- A family of equivariant contraction operations

\[ P(n) \xrightarrow{\xi_{ij}} P(n - 2) \]
Yesterday:

Given a modular operad \( P \), we can construct a presheaf \( NP \in \text{Set}^{\text{Segal}} \):

- \( NP \star_n = P(n) \);
- The Segal maps

\[
NP_G \longrightarrow \prod_{v \in V(G)} NP \star_v
\]

are isomorphisms.

In other words the modular operad compositions and contractions are modelled by:
In this lecture we see examples where the Segal maps:

\[ X_G \longrightarrow \prod_{v \in V(G)} X_{v^*} \]

are weak homotopy equivalences.

Today:

- Introduce space-valued presheaves \( sSet^{U^\text{op}} \); "graphical spaces"

- Describe weak Segal maps;

- Profinite completion of (modular) operads;

- Discuss how variations on the graphical category \( U \) can give genus graded \( \infty \)-modular operads, cyclic operads, etc.

- Open Problems
The category of **graphical spaces** is the category of space-valued \( U \)-presheaves:

\[
\text{Set}^{\mathbf{U}^{\text{op}}}. \quad X : \mathbf{U}^{\text{op}} \to \text{Set}
\]

- \( X_G \): evaluation of \( X \) at \( G \).
- The **representable** presheaf \( U[G] \in \text{Set}^{\mathbf{U}^{\text{op}}} \):

\[
U[G]_H = U(H, G)
\]

for all \( H \) in \( U \).
- \( U[G] \) is an object in \( \text{Set}^{\mathbf{U}^{\text{op}}} \) via

\[
\text{Set}^{\mathbf{U}^{\text{op}}} \hookrightarrow \text{sSet}^{\mathbf{U}^{\text{op}}}.\]
Exercise:

Show $X_G = \text{Set}^{U \text{op}}(U[G], X)$. 
If $G$ a connected graph with $V \neq \emptyset$ we can write $G$ as a coequalizer in $\text{FinSet}^I$:

\[
\begin{array}{c}
\prod_{e \in iE} \uparrow \quad \prod_{v \in V} \star_v \quad \rightarrow \quad G.
\end{array}
\]

Choose an orientation on internal edges.
The **Segal core** of a graph $G$ is the coequalizer in $\text{Set}^{\text{op}}$:

\[
\coprod_{e \in \mathcal{E}} \mathcal{U}[^e] \rightarrow \coprod_{v \in \mathcal{V}} \mathcal{U}[^\star_v] \rightarrow \text{Sc}[G]
\]

in $\text{Set}^{\text{op}}$.

- The embeddings $^\star_v \hookrightarrow G$ induce a map $\text{Sc}[G] \rightarrow \mathcal{U}[G]$.
- If $G = \mathcal{U}$ we set $\text{Sc}[G] = \mathcal{U}[G]$. 
The Segal core is defined so that:

$$\text{Set}^{\text{op}}(\text{Sc}[G], X) = X_G^1 = \lim_{\star_v \leftrightarrow \leftrightarrow \star_w} \left( X_{\star_v} \rightarrow X_{\star_w} \right).$$
The Segal Map:

If \( X_\uparrow = \text{Set}^{\text{op}}(U[\uparrow], X) = * \) then

\[
\text{Set}^{\text{op}}(\text{Sc}[G], X) = \prod_{v \in V(G)} X_{\star v}.
\]

Definition

In the case \( X_\uparrow = * \), the Segal map is given by:

\[
X_G = \text{Set}^{\text{op}}(U[G], X) \rightarrow \text{Set}^{\text{op}}(\text{Sc}[G], X) = \prod_{v \in V(G)} X_{\star v}.
\]

replace these w/ derived mapping.
A presheaf $X \in \text{sSet}^{U^{op}}$ is **Segal** if:

- $X^\uparrow = *$;

- The Segal map

$$\text{map}^h(U[G], X) \longrightarrow \text{map}^h(\text{Sc}[G], X)$$

is a weak equivalence of $\text{sSet}$ for all $G \in U$.

$s\text{Set}^{U^{op}}$ has several model structures

- projective : weak-equivalences $X \stackrel{\sim}{\rightarrow} Y$ if $X_G \stackrel{\sim}{\rightarrow} Y_G$ is a w.equiv. of $\text{sSet}$.

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$\text{Map}_h(A, B) = s\text{Set}^{U^{op}}(A, B) / \simeq$

Strictly speaking not required

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A (dualizable) generalized **Reedy structure** on category $\mathbb{R}$ consists of two wide subcategories $\mathbb{R}^+$ and $\mathbb{R}^-$ together with a degree function $\text{ob}(\mathbb{R}) \to \mathbb{N}$ satisfying:

- non-invertible morphisms in $\mathbb{R}^+$ (respectively $\mathbb{R}^-$) raise (respectively lower degree). Isomorphisms preserve degree.

- $\mathbb{R}^+ \cap \mathbb{R}^- = \text{Iso}(\mathbb{R})$

- Every morphism $f$ factors as $f = gh$ such that $g \in \mathbb{R}^+$ and $h \in \mathbb{R}^-$. Moreover, this factorisation is unique up to isomorphism.

- If $\theta f = f$ for any isomorphism $\theta$ and $f \in \mathbb{R}^-$ then $\theta$ is an identity.

- If $f \theta = f$ for any isomorphism $\theta$ and $f \in \mathbb{R}^+$ then $\theta$ is an identity.
The Reedy Structure of $U$

**Theorem (Berger-Moerdijk)**

If $R$ is a dualizable generalized Reedy category and $\mathcal{E}$ is a nice enough model category, then $\mathcal{E}^{R^{\text{op}}}$ admits a cofibrantly generated model category structure with level-wise weak equivalences.
deg : ob(U) → \mathbb{N} is defined by \text{deg}(G) = |V| + |E|.

**Theorem (Theorem 2.22 HRY1)**

The graphical category \(U\) is a (dualizable) generalised Reedy category. The wide subcategory \(U^-\) is generated by the codegeneracy maps and the wide subcategory \(U^+\) is generated by the inner and outer coface maps.

**Corollary**

The diagram category \(\text{sSet}^{U^{\text{op}}}\) has a model category structure with the Reedy fibrations, Reedy cofibrations, and level-wise weak equivalences.
In Proposition 3.5 of HRY1 we show that Segal cores are cofibrant in the Reedy model structure on $\text{sSet}^{U_{\text{op}}}$. Give an example of a graph $G$ in which the Segal core of $G$ fails to be cofibrant in the projective model structure.
A presheaf $X \in \text{sSet}^{U^{op}}$ is **Segal** if:

- $X_{\uparrow} = \ast$;
- $X$ is Reedy fibrant;
- The Segal map

\[
X_G = \text{map}^h(U[G], X) \xrightarrow{} \text{map}^h(\text{Sc}[G], X) = \prod_{v \in V(G)} X_{\star v}
\]

is a weak equivalence of sSet for all $G \in U$. 
If $P$ is a one-coloured modular operad in $sSet$.

- $NP_{\uparrow} = *$;
- The Segal map

$$NP_G \longrightarrow \prod_{v \in V} NP_{\ast_v}$$

is an isomorphism for every $G$.

$\Rightarrow$ Every one-coloured modular operad gives rise to a Segal modular operad. (up to possible fibrant replacement)
Theorem (Theorem 3.8 HRY1)

The category $\text{sSet}^{\text{op}}$ admits a cofibrantly generated model structure whose fibrant objects are the Segal modular operads.
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The category $sSet^\text{U}^{\text{op}}$ admits a cofibrantly generated model structure whose fibrant objects are the Segal modular operads.

Remark

- One can replace $sSet$ with $\text{Gr}$ or $\text{Cat}$;
- You can also relax the requirement that $X\downarrow = \ast$ and only require that $X\downarrow$ be a contractible space.
Definition
For group $G$, the **profinite completion** of $G$ is the limit

$$\hat{G} = \lim G/N$$

where $N$ runs through all normal subgroups of $G$ of finite index.

Example
The profinite completion of the integers $\hat{\mathbb{Z}} = \lim \mathbb{Z}/\mathbb{Z}_n$ where the limit is taken over the maps $\mathbb{Z}/\mathbb{Z}_n \to \mathbb{Z}/\mathbb{Z}_m$ for $m \mid n$. 

$$\hat{F}_2 = \text{profinite completion of free group on 2-letters}$$
**Definition**
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If $G$ is a **nice enough** groupoid – we can similarly define $\hat{G}$. 

\[ G \text{ has finite objects.} \]
Profinite Completion of Operads

\[
\hat{\mathcal{P}_{\mathcal{A} \mathcal{B}}} = \{ \mathcal{P}_{\mathcal{A} \mathcal{B}}(n) \}
\]

Given a (modular) operad \( P = \{ P(n) \} \) in groupoids is \( \hat{P} = \{ \hat{P}(n) \} \) a (modular) operad in profinite groupoids?
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**Proposition (Horel 17)**

Let $C$ and $D$ be two groupoids (with finite sets of objects). The map

$$\hat{C} \times \hat{D} \to \hat{C} \times \hat{D}$$

is an isomorphism.

$\Rightarrow$

$$\hat{P} = \{\hat{P}(n)\}$$

is a (modular) operad in profinite groupoids.
Theorem (Horel 17)
There is an isomorphism \( \text{Aut}_0(\widehat{\text{PaB}}) \cong \hat{\text{GT}} \).

Theorem (BHR)
There is an isomorphism \( \text{Aut}_0(\widehat{\text{PaRB}}) \cong \hat{\text{GT}} \).

\( \sim \) which fix objects

\( \sim \) parenthesized ribbon braids
Nerve Theorem from yesterday \Rightarrow

\[ \hat{N}P \in \tilde{\text{Gr}}^{\text{op}} : \]

\[ \hat{N}P_G \xrightarrow{\text{ir}} \prod_{v \in V} \hat{N}P_{*v} \] (modular)

\[ \left\{ \begin{array}{c}
\text{operad} \\
\Rightarrow \text{Segal map is an iso.}
\end{array} \right. \]
The functor $\text{Gr} \xrightarrow{B} \text{sSet}$ is symmetric monoidal $\Rightarrow BP = \{BP(n)\}$ is a (modular) operad in profinite spaces.

$$B = \text{classifying space}$$

$$B(PaB)(n) \cong E_2(n)$$

$$B(PaRB)(n) \cong FD_2(n)$$

"closely related"

$\hat{G} T$ act on $\hat{\overline{M}}_{0,n} =$ moduli space of genus $0$ curves w/ $n$ boundaries
The functor \( \text{Gr} \to \text{sSet} \) is symmetric monoidal \( \Rightarrow BP = \{BP(n)\} \) is a (modular) operad in profinite spaces.

**Proposition (BHR)**

Let \( X \) and \( Y \) be two connected spaces whose homotopy groups are **good**. Then the map

\[
\begin{align*}
\hat{X} \times \hat{Y} & \cong X \times Y \\
\end{align*}
\]

is a weak equivalence of profinite spaces.

It's almost never true in general.

Profinite completion of spaces is \( \not\text{good} \).
The functor \( \text{Gr} \xrightarrow{B} \text{sSet} \) is symmetric monoidal \( \Rightarrow BP = \{BP(n)\} \)
is a (modular) operad in profinite spaces.

**Proposition (BHR)**

Let \( X \) and \( Y \) be two connected spaces whose homotopy groups are **good**. Then the map

\[
\hat{X} \times \hat{Y} \to \hat{X} \times \hat{Y}
\]

is a weak equivalence of profinite spaces.

\( \Rightarrow \) If \( BP(n) \) is “nice” for all \( n \), then

\[
\hat{NBPG} \xrightarrow{\sim} \prod_{v \in V} \hat{NB}_{P*_{v}}
\]

and \( \hat{NB} \) is a Segal (modular) operad.
$B\text{PaRB} \simeq \hat{\mathcal{E}}_2$ which is closely related to $\tilde{M}_{0,*}$. So:

**Theorem (BHR)**

There is an isomorphism $\text{Aut}_0(\text{PaRB}) \simeq \widehat{\text{GT}}$.

$\Rightarrow$ GT action on $\tilde{M}_{0,*}$.

* Moerdijk gave a talk on “profinite operad” joint work w/ Blom
Let $G$ be a connected graph:

- A **genus function** for $G$ is a function $l : V(G) \to \mathbb{N}$.
- The **genus** of a pair $(G, l)$ is given by: $g(G) = \beta_1(G) + \sum_{v \in V} l(v)$.
- A pair $(G, l)$ is called **stable** if for every vertex $v$:
  \[ 2l(v) + |\text{nb}(v)| - 2 > 0. \]
The stable graphical category $U_{st}$ has:

- **Objects**: stable graphs $(G, l)$
- **Morphisms**: $(G, l) \rightarrow (G', l')$ are graphical maps $\varphi : G \rightarrow G'$ which make the diagram:

\[
\begin{array}{ccc}
V(G) & \xrightarrow{\varphi_1} & \text{Emb}(G') \\
\downarrow l & & \downarrow l' \\
\mathbb{N} & \xleftarrow{1} & \\
\end{array}
\]

commute.

**Theorem (Remark 4.17 [hry1])**

$U_{st}$ is a generalized Reedy category.
There is model structure on stable graphical spaces in which \( X \in \text{sSet}^{U_{st}^{op}} \) is fibrant if:

- \( X_{\uparrow \downarrow} = *; \)
- \( X \) is Reedy fibrant;
- The Segal map

\[
X_{(G,l)} = \text{map}^h(\text{U}_{st}[G,l], X) \longrightarrow \text{map}^h(\text{Sc}_{st}[G,l], X)
\]

is a weak equivalence, for all \( G \in U_{st}. \)
There is nested sequence of subcategories: \( U_{\text{cyc}} \subset U_0 \subset U \):

- \( U_0 \) are the simply connected graphs in \( U \) which correspond to augmented cyclic operads.
- \( U_{\text{cyc}} \) are the simply connected graphs with non-empty boundary which correspond to cyclic operads.

**Exercise**
The full subcategories \( U_0 \) and \( U_{\text{cyc}} \) are sieves of \( U \). In other words if \( \varphi : G \to T \) is in \( U \) with \( T \in U_0 \) (respectively, \( U_{\text{cyc}} \)) then \( G \in U_0 \) (respectively, \( U_{\text{cyc}} \)).
The category $U_{cyc}$ is related to other categories in the literature:

- **Walde** has a category $\Omega_{cyc}$ which is a non-symmetric version of $U_{cyc}$. That is: $U_{cyc}$ is equivalent to a category $U'_{cyc}$ in which **every object** has a **specified cyclic ordering** and $\Omega_{cyc}$ is the wide subcategory of $U'_{cyc}$ where **maps preserve the ordering**.

- There is another category of Segal cyclic operads $\Xi$ in [hry_cyc]. This category has the same objects as $U_{cyc}$ but slightly different morphisms.
- There are Quillen adjunctions

\[ \text{sSet}^{\text{op}} \cong \text{sSet}^{\text{op}_{\text{cyc}}} \cong \text{sSet}^{\Omega^{\text{op}}}. \]

- Work of Barwick, Hirschhorne and Volic characterizes when

\( F : \mathbb{R} \to \mathcal{S} \) between \textbf{strict} Reedy categories result in Quillen adjunctions between diagram categories. It would be really nice to have such a characterisation for generalized Reedy categories so that all of these comparisons would be straightforward.
- The Segal definition for $\infty$-modular operads will not work (directly) for algebraic modular operads.

- Option 1: A theory of enriched $\infty$-modular operads similar to Chu-Haugsgeng and Chu-Hackney.

- Option 2: Ward that the operad governing modular operads is Koszul (in the setting of groupoid-colored operads). This important result opens the door to effective treatments of strongly homotopy modular operads, in the style of the strongly homotopy operads of van der Laan. This model should be Quillen equivalent to the quasi-modular operad model mentioned in the previous talk (following work of Le Grignou).
simplicial modular operads

\[ \uparrow \quad \text{AE} \]

segal modular operads

\[ \text{Aut}(\text{PaRB}) \simeq G^T \quad \text{"PaRB"} \rightsquigarrow \text{Mon} \]

\[ \text{Gal}(\Omega) \hookrightarrow \Delta \leq G^T \]

\[ \text{Aut}(\hat{\mathcal{M}}_{g,n}) \]
Does a profinite version of $KV$ make sense?

$\hat{GT}_1$ satisfies equations of $KV$?