Lecture 2: A Segal model for ∞-modular operads

Yesterday:

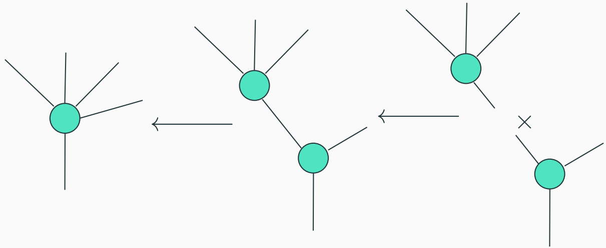
$$\mathrm{ModOp} \xrightarrow[\cong]{N} \mathrm{Set}^{\mathsf{U}^{op}}_{Segal}.$$

Yesterday:

A **modular operad** consists of a collection $\mathsf{P} = \{\mathsf{P}(n)\}$ where:

- Each $\mathsf{P}(n)$ has a Σ_n-action;
- A family of equivariant associative compositions

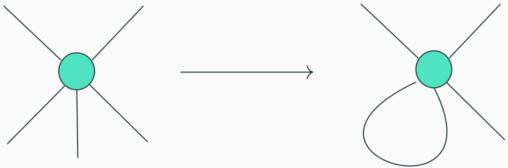
$$\mathsf{P}(n) \times \mathsf{P}(m) \xrightarrow{\circ_{ij}} \mathsf{P}(n+m-2)$$



Yesterday:

- A family of equivariant contraction operations

$$\mathrm{P}(n) \xrightarrow{\xi_{ij}} \mathrm{P}(n-2)$$



Yesterday:

Given a modular operad P, we can construct a presheaf $N\mathrm{P} \in \mathrm{Set}^{\mathrm{U}^{op}}_{Segal}$:

- $N\mathrm{P}_{\bigstar_n} = \mathrm{P}(n)$;
- The Segal maps

$$N\mathrm{P}_G \longrightarrow \prod_{v \in V(G)} N\mathrm{P}_{\bigstar_v}$$

  are isomorphisms.

In other words the modular operad compositions and contractions are modelled by:

$$N\mathrm{P}_{\bigstar_G} \longleftarrow N\mathrm{P}_G \xrightarrow{\cong} \prod_{v \in V(G)} N\mathrm{P}_{\bigstar_v}$$

Goal:

In this lecture we see examples where the Segal maps:

$$X_G \longrightarrow \prod_{v \in V(G)} X_{\bigstar_v}$$

are weak homotopy equivalences.

Today:

- Introduce space-valued presheaves $\mathrm{sSet}^{\mathrm{U}^{op}}$;
- Describe weak Segal maps;
- Profinite completion of (modular) operads;
- Discuss how variations on the graphical category U can give genus graded ∞-modular operads, cyclic operads, etc.
- Open Problems

Graphical Spaces

The category of **graphical spaces** is the category of space-valued U-presheaves:

$$\mathsf{sSet}^{\mathsf{U}^{op}}.$$

- X_G: evaluation of X at G.
- The **representable** presheaf $\mathsf{U}[G] \in \mathsf{Set}^{\mathsf{U}^{op}}$:

  $$\mathsf{U}[G]_H = \mathsf{U}(H, G)$$

  for all H in U.
- $\mathsf{U}[G]$ is an object in $\mathsf{sSet}^{\mathsf{U}^{op}}$ via

  $$\mathsf{Set}^{\mathsf{U}^{op}} \hookrightarrow \mathsf{sSet}^{\mathsf{U}^{op}}.$$

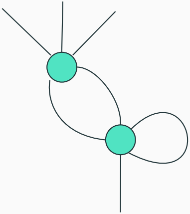
Exercise:

Show $X_G = \mathsf{Set}^{\mathsf{U}^{op}}(\mathsf{U}[G], X)$.

Segal Core:

If G a connected graph with $V \neq \emptyset$ we can write G as a coequalizer in $\mathrm{FinSet}^{\mathcal{I}}$:

$$\coprod_{e \in \mathrm{iE}} \updownarrow \rightrightarrows \coprod_{v \in V} \bigstar_v \longrightarrow G.$$



Segal Core:

The **Segal core** of a graph G is the coequalizer in $\mathsf{Set}^{\mathsf{U}^{op}}$:

$$\coprod_{e \in \mathrm{iE}} \mathsf{U}[\updownarrow] \rightrightarrows \coprod_{v \in V} \mathsf{U}[\bigstar_v] \longrightarrow \mathsf{Sc}[G]$$

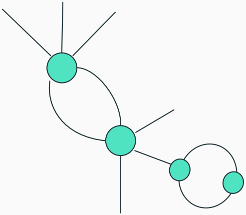
in $\mathsf{Set}^{\mathsf{U}^{op}}$.

- The embeddings $\bigstar_v \hookrightarrow G$ induce a map $\mathsf{Sc}[G] \to \mathsf{U}[G]$.
- If $G = \updownarrow$ we set $\mathsf{Sc}[G] = \mathsf{U}[G]$.

Segal Core:

The Segal core is defined so that:

$$\mathsf{Set}^{\mathsf{U}^{op}}(\mathsf{Sc}[G], X) = X_G^1 = \lim_{\bigstar_v \leftarrow \updownarrow \rightarrow \bigstar_w} \left(X_{\bigstar_v} \rightarrow X_{\updownarrow} \leftarrow X_{\bigstar_w} \right).$$



The Segal Map:

If $X_{\updownarrow} = \mathsf{Set}^{\mathsf{U}^{op}}(\mathsf{U}[\updownarrow], X) = *$ then

$$\mathsf{Set}^{\mathsf{U}^{op}}(\mathsf{Sc}[G], X) = \prod_{v \in V(G)} X_{\bigstar_v}.$$

Definition
In the case $X_{\updownarrow} = *$, the Segal map is given by:

$$X_G = \mathsf{Set}^{\mathsf{U}^{op}}(\mathsf{U}[G], X) \longrightarrow \mathsf{Set}^{\mathsf{U}^{op}}(\mathsf{Sc}[G], X) = \prod_{v \in V(G)} X_{\bigstar_v}.$$

Segal Modular Operads: A first definition

A presheaf $X \in \mathsf{sSet}^{\mathsf{U}^{op}}$ is **Segal** if:

- $X_{\updownarrow} = *$;
- The Segal map

$$\mathrm{map}^h(\mathsf{U}[G], X) \longrightarrow \mathrm{map}^h(\mathsf{Sc}[G], X)$$

  is a weak equivalence of sSet for all $G \in \mathsf{U}$.

The Reedy Structure of $\mathbb{U}$

A (dualizable) generalized **Reedy structure** on category $\mathbb{R}$ consists of two wide subcategories $\mathbb{R}^+$ and $\mathbb{R}^-$ together with a degree function $\mathrm{ob}(\mathbb{R}) \to \mathbb{N}$ satisfying:

- non-invertible morphisms in $\mathbb{R}^+$ (respectively $\mathbb{R}^-$) raise (respectively lower degree). Isomorphisms preserve degree.
- $\mathbb{R}^+ \bigcap \mathbb{R}^- = \mathrm{Iso}(\mathbb{R})$
- Every morphism f factors as $f = gh$ such that $g \in \mathbb{R}^+$ and $h \in \mathbb{R}^-$. Moreover, this factorisation is unique up to isomorphism.
- If $\theta f = f$ for any isoorphism θ and $f \in \mathbb{R}^-$ then θ is an identity.
- If $f\theta = f$ for any isoorphism θ and $f \in \mathbb{R}^+$ then θ is an identity.

The Reedy Structure of $\mathbb{U}$

Theorem (Berger-Moerdijk)
If $\mathbb{R}$ is a dualizable generalized Reedy category and $\mathcal{E}$ is a nice enough model category, then $\mathcal{E}^{\mathbb{R}^{op}}$ admits a cofibrantly generated model category structure with level-wise weak equivalences.

The Reedy Structure of U

$\deg : \mathrm{ob}(\mathsf{U}) \to \mathbb{N}$ is defined by $\deg(G) = |V| + |iE|$.

Theorem (Theorem 2.22 HRY1)
The graphical category U is a (dualizable) generalised Reedy category. The wide subcategory U^- is generated by the codegeneracy maps and the wide subcategory U^+ is generated by the inner and outer coface maps.

Corollary
The diagram category $\mathrm{sSet}^{\mathsf{U}^{op}}$ has a model category structure with the Reedy fibrations, Reedy cofibrations, and level-wise weak equivalences.

Exercise

In Proposition 3.5 of HRY1 we show that Segal cores are cofibrant in the Reedy model structure on $\mathrm{sSet}^{\mathsf{U}^{op}}$. Give an example of a graph G in which the Segal core of G fails to be cofibrant in the projective model structure.

Segal Modular Operads

A presheaf $X \in \mathrm{sSet}^{\mathsf{U}^{op}}$ is **Segal** if:

- $X_{\updownarrow} = *$;
- X is Reedy fibrant;
- The Segal map

$$X_G = \mathrm{map}^h(\mathsf{U}[G], X) \longrightarrow \mathrm{map}^h(\mathrm{Sc}[G], X) = \prod_{v \in V(G)} X_{\bigstar_v}$$

is a weak equivalence of sSet for all $G \in \mathsf{U}$.

Segal Modular Operads

If P is a one-coloured modular operad in sSet.

- $N\mathsf{P}_{\updownarrow} = *$;
- The Segal map

$$N\mathsf{P}_G \longrightarrow \prod_{v \in V} N\mathsf{P}_{\star_v}$$

  is an isomorphism for every G.

$\Rightarrow$ Every one-coloured modular operad gives rise to a Segal modular operad.

Classifying Segal Modular Operads

Theorem (Theorem 3.8 HRY1)
The category $\mathrm{sSet}^{\mathsf{U}^{op}}$ *admits a cofibrantly generated model structure whose fibrant objects are the Segal modular operads.*

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Remark

- One can replace sSet with Gr or Cat;
- You can also relax the requirement that $X_{\updownarrow} = *$ and only require that $X_{\updownarrow}$ be a contractible space.

Profinite Completion of Operads

Definition
For group G, the **profinite completion** of G is the limit

$$\widehat{G} = \lim G/N$$

where N runs through all normal subgroups of G of finite index.

Example
The profinite completion of the integers $\widehat{\mathbb{Z}} = \lim \mathbb{Z}/\mathbb{Z}_n$ where the limit is taken over the maps $\mathbb{Z}/\mathbb{Z}_n \to \mathbb{Z}/\mathbb{Z}_m$ for $m \mid n$.

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If G is a nice enough **groupoid** – we can similarly define $\widehat{G}$.

Profinite Completion of Operads

Given a (modular) operad $\mathsf{P} = \{\mathsf{P}(n)\}$ in groupoids is $\widehat{\mathsf{P}} = \{\widehat{\mathsf{P}(n)}\}$ a (modular) operad in profinite groupoids?

Profinite Completion of Operads

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Proposition (Horel 17)
Let C and D be two groupoids (with finite sets of objects). The map

$$\widehat{C \times D} \to \widehat{C} \times \widehat{D}$$

is an isomorphism.

$\Rightarrow$

$$\widehat{\mathsf{P}} = \{\widehat{\mathsf{P}(n)}\}$$

is a (modular) operad in profinite groupoids.

Profinite Completion of Operads

Theorem (Horel 17)
There is an isomorphism $\mathrm{Aut}_0(\widehat{\mathsf{PaB}}) \cong \mathsf{GT}$.

Theorem (BHR)
There is an isomorphism $\mathrm{Aut}_0(\widehat{\mathsf{PaRB}}) \cong \mathsf{GT}$.

Profinite Completion of Operads

Nerve Theorem from yesterday $\Rightarrow$

$$N\widehat{\mathsf{P}} \in \mathbf{Gr}^{\mathsf{U}^{op}} :$$

$$N\widehat{\mathsf{P}}_G \xrightarrow{\cong} \prod_{v \in V} N\widehat{\mathsf{P}}_{\star_v}$$

for every graph G.

Profinite Completion of Operads

The functor $\mathsf{Gr} \xrightarrow{B} \mathsf{sSet}$ is symmetric monoidal $\Rightarrow BP = \{BP(n)\}$ is a (modular) operad in profinite spaces.

Profinite Completion of Operads

The functor $\mathrm{Gr} \xrightarrow{B} \mathrm{sSet}$ is symmetric monoidal $\Rightarrow B\mathrm{P} = \{B\mathrm{P}(n)\}$ is a (modular) operad in profinite spaces.

Proposition (BHR)
*Let X and Y be two connected spaces whose homotopy groups are **good**. Then the map*

$$\widehat{X \times Y} \longrightarrow \widehat{X} \times \widehat{Y}$$

is a weak equivalence of profinite spaces.

Profinite Completion of Operads

The functor $\mathrm{Gr} \xrightarrow{B} \mathrm{sSet}$ is symmetric monoidal $\Rightarrow B\mathrm{P} = \{B\mathrm{P}(n)\}$ is a (modular) operad in profinite spaces.

Proposition (BHR)
Let X and Y be two connected spaces whose homotopy groups are ***good****. Then the map*

$$\widehat{X \times Y} \longrightarrow \widehat{X} \times \widehat{Y}$$

is a weak equivalence of profinite spaces.

$\Rightarrow$ If $B\mathrm{P}(n)$ is "nice" for all n, then

$$N\widehat{B\mathrm{P}}_G \xrightarrow{\simeq} \prod_{v \in V} N\widehat{B\mathrm{P}}_{\star_v}$$

and $N\widehat{B\mathrm{P}}$ is a Segal (modular) operad.

Profinite Completion of Operads

$B\mathsf{PaRB} \simeq \mathsf{fE}_2$ which is closely related to $\bar{\mathcal{M}}_{0,*}$. So :

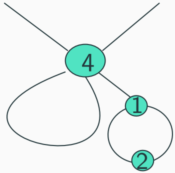
Theorem (BHR)
There is an isomorphism $\mathrm{Aut}_0(\widehat{\mathsf{PaRB}}) \cong \mathsf{GT}$.

$\Rightarrow$ GT action on $\bar{\mathcal{M}}_{0,*}$.

Variations on the graphical category $\mathbb{U}$: Genus Graded Modular Operads

Let G be a connected graph:

- A **genus function** for G is a function $l : V(G) \to \mathbb{N}$.
- The **genus** of a pair (G, l) is given by: $g(G) = \beta_1(G) + \Sigma_{v \in V} l(v)$.
- A pair (G, l) is called **stable** if for every vertex v: $2l(v) + |\,\mathrm{nb}(v)| - 2 > 0$.



The stable graphical category

The stable graphical category U_{st} has:

- Objects: stable graphs (G, l)
- Morphisms: $(G, l) \to (G', l')$ are graphical maps $\varphi : G \to G'$ which make the diagram:

$$\begin{array}{ccc} V(G) & \xrightarrow{\varphi_1} & \mathrm{Emb}(G') \\ & {\scriptstyle l}\searrow \quad \swarrow{\scriptstyle l'} & \\ & \mathbb{N} & \end{array}$$

  commute.

Theorem (Remark 4.17 [hry1])
U_{st} *is a generalizsed Reedy category.*

Segal Modular Operads

There is model structure on stable graphical spaces in which $X \in \mathsf{sSet}^{\mathsf{U}_{st}^{op}}$ is fibrant if:

- $X_{\updownarrow} = *$;
- X is Reedy fibrant;
- The Segal map

$$X_{(G,I)} = \mathrm{map}^h(\mathsf{U}_{st}[(G,I)], X) \longrightarrow \mathrm{map}^h(\mathsf{Sc}_{st}[(G,I)], X)$$

is a weak equivalence, for all $G \in \mathsf{U}_{st}$.

Cyclic Operads

There is nested sequence of subcategories: $\mathsf{U}_{cyc} \subset \mathsf{U}_0 \subset \mathsf{U}$:

- U_0 are the simply connected graphs in U which correspond to augmented cyclic operads.
- U_{cyc} are the simply connected graphs with non-empty boundary which correspond to cyclic operads.

Exercise
The full subcategories U_0 and U_{cyc} are sieves of U. In other words if $\varphi : G \to T$ is in U with $T \in \mathsf{U}_0$ (respectively, U_{cyc}) then $G \in \mathsf{U}_0$ (respectively, U_{cyc}).

Related Work and Future Directions

The category U_{cyc} is related to other categories in the literature:

- Walde has a category Ω_{cyc} which is a non-symmetric version of U_{cyc}. That is: U_{cyc} is equivalent to a category U'_{cyc} in which every object has a specified cyclic ordering and Ω_{cyc} is the wide subcategory of U'_{cyc} where maps preserve the ordering.
- There is another category of Segal cyclic operads Ξ in [**hry_cyc**]. This category has the same objects as U_{cyc} but slightly different morphisms.

Related Work and Future Directions

- There are Quillen adjunctions

$$\mathrm{sSet}^{\Xi^{op}} \leftrightarrows \mathrm{sSet}^{\mathrm{U}_{cyc}^{op}} \leftrightarrows \mathrm{sSet}^{\Omega^{op}}.$$

- Work of Barwick, Hirschhorne and Volic characterizes when $F : \mathbb{R} \to \mathbb{S}$ between **strict** Reedy categories result in Quillen adjunctions between diagram categories. It would be really nice to have such a characterisation for generalized Reedy categories so that all of these comparisons would be straightforward.

Related Work and Future Directions

- The Segal defnition for ∞-modular operads will not work (directly) for algebraic modular operads.
- Option 1: A theory of enriched ∞-modular operads similar to Chu-Haugseng and Chu-Hackney.
- Option 2: Ward that the operad governing modular operads is Koszul (in the setting of groupoid-colored operads). This important result opens the door to effective treatments of strongly homotopy modular operads, in the style of the strongly homotopy operads of van der Laan. This model should be Quillen equivalent to the quasi-modular operad model mentioned in the previous talk (following work of Le Grignou).