Lecture 3: Grothendieck-Teichmüller Theory and Modular Operads

A really fast introduction to a lot of cool math:

- Let $Gal(\mathbb{Q})$ denote the absolute Galois group of $\mathbb{Q}$.
- The "Grothendieck part" of Grothendieck-Teichmüller theory $\Rightarrow$ identify $g \in Gal(\mathbb{Q})$ with a pair

$$(\chi(g), f_g) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$$

- $\chi(g)$ is the just the **cyclotomic character**
- $\widehat{F}_2$ is the "Teichmüller part" of Grothendieck-Teichmüller theory – $\widehat{F}_2 = \pi_1(\mathcal{M}_{0,4})$

$\Rightarrow$ **Idea:** study the geometric actions of $Gal(\mathbb{Q})$ on $\mathcal{M}_{g,n}$ to find necessary and sufficient conditions for an element of $f \in \widehat{F}_2'$ to come from a $g \in Gal(\mathbb{Q})$.

Profinite $\widehat{\mathrm{GT}}$

Notation: For any homomorphism of profinite groups

$$\widehat{F}_2 \longrightarrow G$$

$$(x, y) \longmapsto (a, b)$$

we write $f(a, b)$ for the image of any $f \in \widehat{F}_2$. For example:

- Given $id : \widehat{F}_2 \to \widehat{F}_2$, we have $f = f(x, y)$;
- Given the map $\widehat{F}_2 \to \widehat{F}_2$ which swaps generators x and y we have $f \mapsto f(y, x)$.

Profinite $\widehat{\mathrm{GT}}$

The **Grothendieck-Teichmüller group** $\widehat{\mathrm{GT}}$ is the group of pairs

$$(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$$

satsfying the property that

$$x \mapsto x^\lambda \quad \text{and} \quad y \mapsto f^{-1} y^\lambda f$$

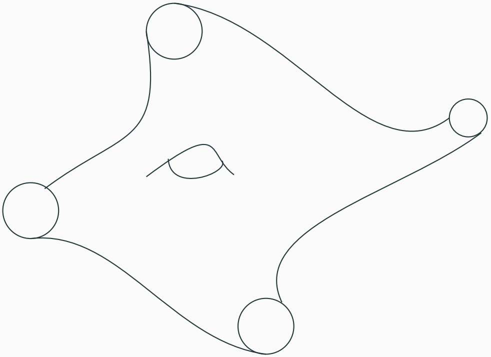
induce an automorphism of $\widehat{F}_2$ and :

(I) $f(x,y)f(y,x) = 1$,
(II) $f(x,y)x^m f(z,x)z^m f(y,z)y^m = 1$ where $xyz = 1$ and $m = (\lambda-1)/2$,
(III) $f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51})f(x_{12}, x_{23}) = 1$ in $\Gamma_{0,5}$ where x_{ij} is a Dehn twist of boundaries i and j.

Theorem (Ihara)
There is an injection $Gal(\mathbb{Q}) \hookrightarrow \widehat{\mathrm{GT}}$.

Points of $\mathcal{M}_{g,n}$ are isomorphism classes of $\Sigma_{g,n}$.

Let $\Sigma_{g,n}$ denote a Riemann surface with *n*-boundaries:



Studying the Moduli Space via Surfaces

The **profinite mapping class group** of a surface Σ:

$$\widehat{\Gamma}_{g,n} = \pi_0 \mathrm{Diff}^+(\Sigma, \partial)$$

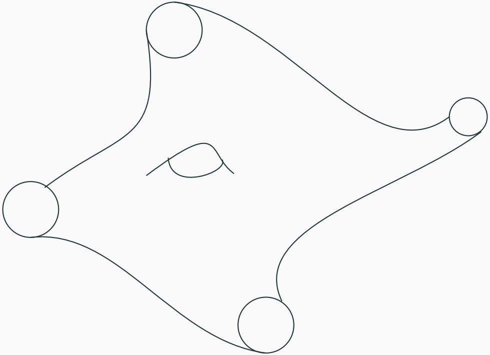
can be identified with with the fundamental group of $\pi_1(\mathcal{M}_{g,n}, \Sigma)$:

$$\widehat{\Gamma}_{g,n} \cong \pi_1(\mathcal{M}_{g,n}) \cong \pi_0 \mathrm{Diff}^+(\Sigma, \partial).$$

Studying the Moduli Space via Surfaces

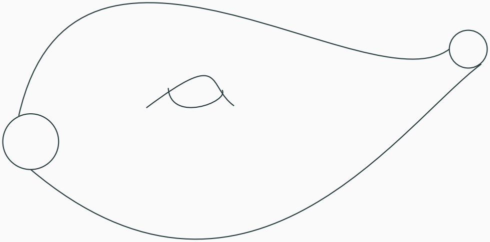
Theorem (Hatcher-Thurston)
The mapping class group has a presentation

$$\Gamma_{g,n} = \langle \alpha_1, \ldots, \alpha_k \mid (C), (B), (D), (L) \rangle .$$



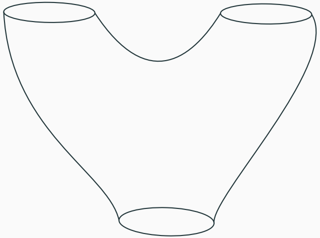
Pants Decompositions

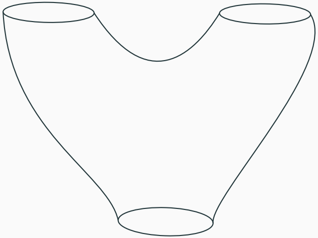
A **pants decomposition** of $\Sigma_{g,n}$ is a collection of simple closed curves that cuts $\Sigma_{g,n}$ into *pairs of pants* (i.e. $\Sigma_{0,3}$).



Quilted Pants Decompositions

We actually want **quilted** pants decompositions.





Quilted Pants Decompositions

We actually want **quilted** pants decompositions.

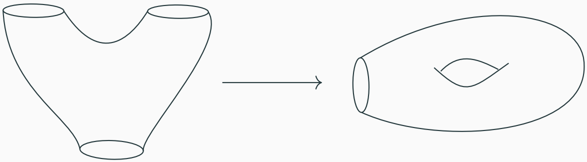
Definition
A **quilt** consists of:

- 2 marked points, called *vertices*, for every circle in the decomposition;
- 3 disjoint lines between the vertices for each pair of pants;

Fact: A quilt Q on $\Sigma_{g,n}$ is completely determined by the underlying pants decomposition P together with the vertices and seams.

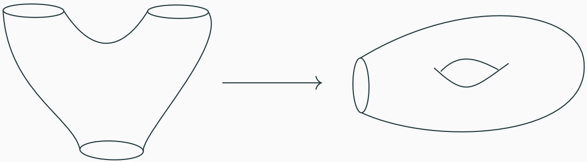
Studying the Moduli Space via Surfaces

Notice that a **pants decomposition** looks like the result of an operadic composition.



Studying the Moduli Space via Surfaces

A quilted **pants decomposition** can also be determined by (modular) operadic composition.



A modular operad of Seamed Surfaces

The goupoid $\mathcal{S}_{g,n}$:

- objects are surfaces $Q/P := (\Sigma_{g,n}, P, Q)$ together with a "atomic" quilt decomposition;
- morphisms are $\pi_0\mathrm{Diff}^+(\Sigma_{g,n}, \partial, \sigma)$.

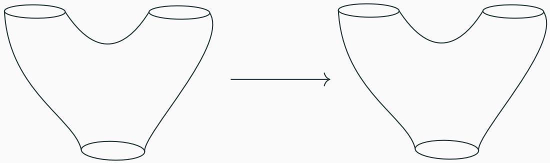
Σ_n acts freely on $\mathcal{S}_{g,n}$ by permuting the labels of boundaries $\Rightarrow$

$$B\mathcal{S}_{g,n} \simeq B\Gamma_{g,n}.$$

A modular operad of Seamed Surfaces

The morphisms in $\mathcal{S}_{g,n}$ are a composite of three types of elementary morphisms:

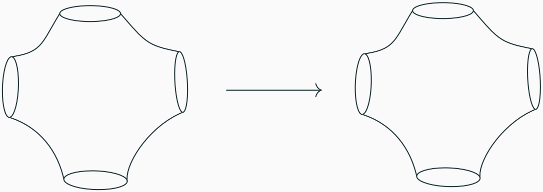
- $\frac{1}{2}$ – Dehn twists



A modular operad of Seamed Surfaces

The morphisms in $\mathcal{S}_{g,n}$ are a composite of three types of elementary morphisms:

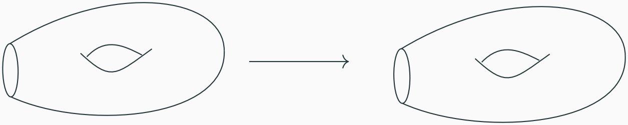
- *A*-moves



A modular operad of Seamed Surfaces

The morphisms in $\mathcal{S}_{g,n}$ are a composite of three types of elementary morphisms:

- S-moves



A modular operad of Seamed Surfaces

The elementary morphisms in $\mathcal{S}_{g,n}$ satisfy 6 important relations such as:

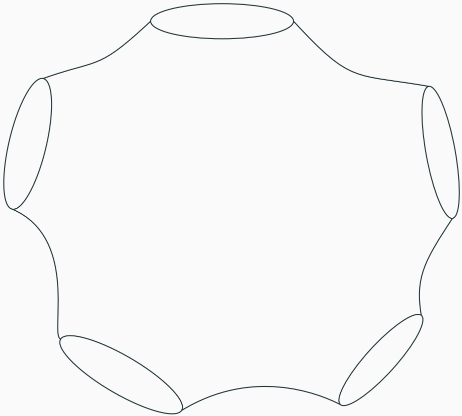
$$A_{\beta_{\mathbf{3}},\beta_{\mathbf{4}}} A_{\beta_{\mathbf{1}},\beta_{\mathbf{2}}} A_{\beta_{\mathbf{4}},\beta_{\mathbf{5}}} A_{\beta_{\mathbf{2}},\beta_{\mathbf{3}}} A_{\beta_{\mathbf{5}},\beta_{\mathbf{1}}} = 1 \tag{5A}$$

$$S_{\beta_{\mathbf{1}},\beta_{\mathbf{2}}} S_{\beta_{\mathbf{2}},\beta_{\mathbf{3}}} S_{\beta_{\mathbf{3}},\beta_{\mathbf{1}}} = 1 \tag{3S}$$

$$A_{\alpha_{\mathbf{3}},\epsilon_{\mathbf{3}}} S_{\alpha_{\mathbf{1}},\alpha_{\mathbf{2}}} A_{\epsilon_{\mathbf{3}},\epsilon_{\mathbf{2}}} A_{\epsilon_{\mathbf{2}},\epsilon_{\mathbf{1}}} S_{\alpha_{\mathbf{2}},\alpha_{\mathbf{3}}} A_{\epsilon_{\mathbf{1}},\alpha_{\mathbf{1}}} = 1. \tag{6AS}$$

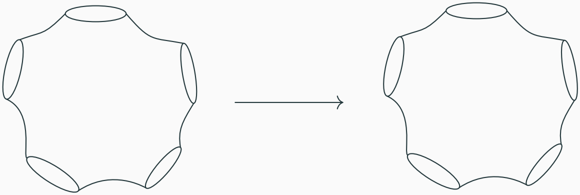
Relations on $\mathcal{S}_{g,n}$

The A-move $A_{\beta_{\mathbf{3}},\beta_{\mathbf{4}}}$ is referring to the fact all possible pants decompositions of $\Sigma_{0,5}$ are:



Relations on $\mathcal{S}_{g,n}$

The A-move A_{β_3,β_4} is a diffeomorphism from the pants decomposition containing β_3 to the decomposition containing β_4:



A modular operad of Seamed Surfaces

Operations:

$$\mathcal{S}_{g,n} \times_{ij} \mathcal{S}_{h,k} \xrightarrow{\circ_{ij}} \mathcal{S}_{g+h,n+k-2}$$

and

$$\mathcal{S}_{g,n} \xrightarrow{\xi_{ij}} \mathcal{P}_{g+1,n-2}$$

can be defined on objects by gluing surfaces and on morphisms as the “combination" of the maps on the subsurfaces.

These are well-defined, associative operations and thus

$$\mathcal{S} = \{\mathcal{S}_{g,n}\}$$

assembles into a **modular operad in groupoids**.

The genus 0 case

Let $\Sigma_{0,n+1}$ denote an oriented surface of genus 0 with $n+1$ boundary components. We choose one of these boundary components and call it *marked* and we say the other n components are *free*.

Definition (Tillmann)
The groupoid $\mathcal{S}_{0,n+1}$:

- objects are $\Sigma_{0,n+1}$ together with a "atomic" pants decomposition;
- morphisms are $\pi_0\mathrm{Diff}^+(\Sigma_{g,n+1}, \partial, \sigma)$.

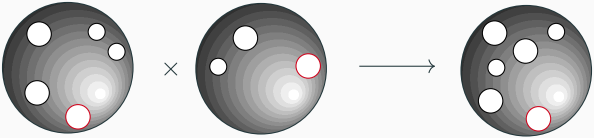
As before: Σ_n acts freely on the free boundaries $\mathcal{S}_{0,n+1}$ and

$$B\mathcal{S}_{0,n+1} \simeq B\Gamma_{0,n}.$$

The genus 0 case

One obtains **operad** composition maps

$$\mathcal{S}_{0,n+1} \times \mathcal{S}_{0,m+1} \xrightarrow{\circ_i} \mathcal{S}_{0,n+m}$$



Fact: The resulting operad in groupoids $\mathcal{S} = \{\mathcal{S}_{0,n+1}\}$ is an operad "equivalent" to $\mathrm{PaRB} = \{\mathrm{PaRB}(n)\}$.

The genus 0 case

Yesterday:

Proposition (Horel)
Let C and D be two groupoids (with finite sets of objects). The map

$$\widehat{C \times D} \to \widehat{C} \times \widehat{D}$$

is an isomorphism.

$\Rightarrow$ The profinite completion $\widehat{\mathcal{S}} = \{\widehat{\mathcal{S}}_{0,n+1}\}$ is an operad in profinite groupoids.

An action of $\widehat{\mathsf{GT}}$

Proposition (BHR)
There's an action of $\widehat{\mathsf{GT}}$ *on* $\widehat{\mathcal{S}}$*.*

In the paper:

Proposition (BHR)

$$\widehat{\mathsf{GT}} \cong \mathsf{End}_0(\widehat{\mathsf{PaRB}})$$

An action of $\widehat{\mathsf{GT}}$

So let's describe a homomorphism

$$\widehat{\mathsf{GT}} \longrightarrow \mathsf{End}_0(\widehat{\mathcal{S}}).$$

An action of $\widehat{\mathrm{GT}}$

So let's describe a homomorphism

$$\widehat{\mathrm{GT}} \longrightarrow \mathrm{End}_0(\widehat{\mathcal{S}}).$$

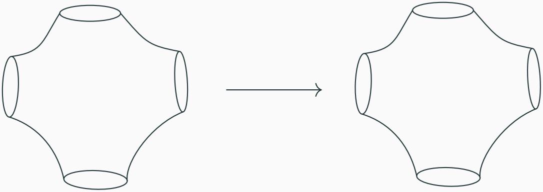
Given a pair $(\lambda, f) \in \widehat{\mathrm{GT}}$:

$$(\lambda, f) \longrightarrow \begin{cases} a_{\frac{1}{2}} \mapsto a_{\frac{1}{2}}^{\lambda} \\ A_{\alpha,\beta} \mapsto A_{\alpha,\beta} \cdot f(a, b) a^{N(\lambda-1)/2} \end{cases}$$

where N is some integer (a "twist").

An action of $\widehat{\mathrm{GT}}$

Recall that the A-move $A_{\alpha,\beta}$ is the diffeomorphism pictured below.



An action of $\widehat{\mathrm{GT}}$

The relations on morphisms in $\mathcal{S}_{0,n+1}$ will imply that this action is well defined.

$$A_{\beta_3,\beta_4}A_{\beta_1,\beta_2}A_{\beta_4,\beta_5}A_{\beta_2,\beta_3}A_{\beta_5,\beta_1} = 1 \tag{5A}$$

$$A_{\beta_3,\beta_4}A_{\beta_1,\beta_2}A_{\beta_4,\beta_5}A_{\beta_2,\beta_3}A_{\beta_5,\beta_1} \mapsto$$
$$A_{\beta_3,\beta_4}f(b_1, b_2)b_1^{N(\lambda-1)/2} \ldots A_{\beta_5,\beta_1}f(b_5, b_1)b_5^{N(\lambda-1)/2}$$

An action of $\widehat{\mathrm{GT}}$

We wont explain it here, but the integers N in this situation all go to 0:

$$A_{\beta_3,\beta_4} f(b_3, b_4) b_3^{N(\lambda-1)/2} \dots A_{\beta_5,\beta_1} f(b_5, b_1) b_5^{N(\lambda-1)/2} = A_{\beta_3,\beta_4} f(b_3, b_4) \dots A_{\beta_5,\beta_1} f(b_5, b_1)$$

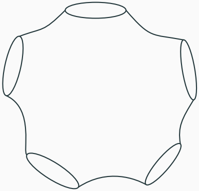
We can pull the values words in F_2 to the left:

$$A_{\beta_3,\beta_4} f(b_3, b_4) \dots A_{\beta_5,\beta_1} f(b_5, b_1) = (A_{\beta_3,\beta_4} \dots A_{\beta_5,\beta_1}) \cdot (f(b_3, b_4) \dots f(b_5, b_1))$$

An action of $\widehat{\mathrm{GT}}$

Reducing by an application of the relation (5A) and we have

$$f(b_3, b_4)f(b_1, b_2)f(b_4, b_5)f(b_2, b_3)f(b_5, b_1) = 1$$



After some translation we arrive at:

(III) $f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51})f(x_{12}, x_{23}) = 1$

in $\widehat{\mathrm{GT}}$

The genus 0 case

Proposition (BHR)

$$\widehat{\mathsf{GT}} \cong \pi_0 \operatorname{map}^h(N\widehat{\mathcal{S}}, N\widehat{\mathcal{S}})$$

The genus 0 case

Proposition (BHR)

$$\widehat{\mathsf{GT}} \cong \pi_0 \operatorname{map}^h(N\widehat{\mathcal{S}}, N\widehat{\mathcal{S}})$$

As we mentioned yesterday $B\mathcal{S}_{0,n+1}$ are nice enough spaces so that:

Theorem (BHR)

$$\widehat{\mathsf{GT}} \cong \pi_0 \operatorname{map}^h(B\widehat{N\mathcal{S}}, B\widehat{N\mathcal{S}})$$

The genus 0 case

The spaces $B\mathcal{S}_{0,n+1} \simeq B\mathrm{PaRB}(n) \simeq \mathrm{FD}(n)$ are very closely related to the moduli space $\bar{\mathcal{M}}_{0,n}$:

Theorem (Drummond-Cole)
There's a homotopy pushout of operads :

$$\begin{array}{ccc} S^1 & \longrightarrow & \mathrm{FD} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \bar{\mathcal{M}}_{0,\bullet+1} \end{array}$$

The moduli spaces $\bar{\mathcal{M}}_{0,n}$ are "nice enough" spaces so that we have :

Theorem (BHR)
The dendroidal profinite space $(N\bar{\mathcal{M}}_{0,\bullet+1})$ *is an* ∞*-operad.*

The genus 0 case

Theorem (BHR)
There is a non-trivial action of $\widehat{\mathsf{GT}}$ *on the* ∞*-operad* $(N\bar{\mathcal{M}}_{0,\bullet+1})$.

The cyclic genus 0 case

Nothing we have said today specifically relied on that fixed boundary, and so we expect the cyclic version:

Proposition (In progress)

$$\widehat{\mathsf{GT}} \cong \pi_0 \operatorname{map}^h(N\widehat{\mathcal{S}}, N\widehat{\mathcal{S}})$$

Groups Related to $\widehat{\mathrm{GT}}$: The higher genus case

Definition (Nakamura-Schneps)
Let Λ denote the group of pairs

$$(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$$

which satisfy the property that

$$x \mapsto x^\lambda \quad \text{and} \quad y \mapsto f^{-1} y^\lambda f$$

induce an automorphism of $\widehat{F}_2$ and relations $(I) - (III)$ of $\widehat{\mathrm{GT}}$ and :

(IV)

$$f(e_1, a_1) a_3^{-8\rho_2} f(a_2^2, a_3^2)(a_3 a_2 a_3)^{2m} f(e_2, e_1) e_2^{2m} f(e_3, e_2) \\ a_2^{-2m}(a_1 a_2 a_1)^{2m} f(a_1^2, a_2^2) a_1^{8\rho_2} f(a_3, e_3) = 1$$

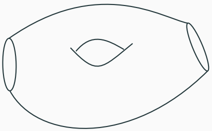
where a_1, a_2, a_3, e_1, e_2 are Dehn twists in $\Gamma_{1,2}$, $m = (\lambda - 1)/2$ and ρ_2 is an integer.

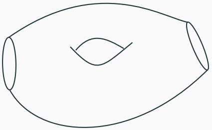
Groups Related to $\widehat{\mathrm{GT}}$: The higher genus case

(IV)

$$f(e_1, a_1)a_3^{-8\rho_2} f(a_2^2, a_3^2)(a_3 a_2 a_3)^{2m} f(e_2, e_1) e_2^{2m} f(e_3, e_2)$$
$$a_2^{-2m}(a_1 a_2 a_1)^{2m} f(a_1^2, a_2^2) a_1^{8\rho_2} f(a_3, e_3) = \quad 1$$

where a_1, a_2, a_3, e_1, e_2 are Dehn twists in $\Gamma_{1,2}$, $m = (\lambda - 1)/2$ and ρ_2 is an integer.





Groups Related to $\widehat{\mathrm{GT}}$: The higher genus case

Proposition (BR)
There is an action

$$\Lambda \longrightarrow \mathrm{End}_0(\widehat{\mathcal{S}}).$$

Given a pair $(\lambda, f) \in \Lambda$:

$$(\lambda, f) \longrightarrow \begin{cases} a_{\frac{1}{2}} \mapsto a_{\frac{1}{2}}^{\lambda} \\ A_{\alpha,\beta} \mapsto A_{\alpha,\beta} \cdot f(a,b) a^{N(\lambda-1)/2} \\ S_{\alpha,\beta} \mapsto S_{\alpha,\beta} \cdot (aba)^{\lambda-1} b^{N(\lambda-1)/2-8\rho_2} f(a^2, b^2) a^{8\rho_2} \end{cases}$$

where N is some integer (a "twist") and ρ_2 is a different integer.

Let's check this is well-defined

Recall that the S-move $S_{\alpha,\beta}$ is the diffeomorphism:

Let's check this is well-defined

We just need to check that the action

$$S_{\alpha,\beta} \mapsto S_{\alpha,\beta} \cdot (aba)^{\lambda-1} b^{N(\lambda-1)/2-8\rho_2} f(a^2, b^2) a^{8\rho_2}$$

interacts well with the relations on the morphisms of the groupoids $\mathcal{S}_{g,n}$, eg:

$$A_{\alpha_3,\epsilon_3} S_{\alpha_1,\alpha_2} A_{\epsilon_3,\epsilon_2} A_{\epsilon_2,\epsilon_1} S_{\alpha_2,\alpha_3} A_{\epsilon_1,\alpha_1} = 1 \qquad \text{(6AS)}$$

Let's check this is well-defined

$$A_{\alpha_3,\epsilon_3} S_{\alpha_1,\alpha_2} \ldots A_{\epsilon_1,\alpha_1} \mapsto$$

$$A_{\alpha_3,\epsilon_3} f(a_3, e_3) a_3^{N(\lambda-1)/2} S_{\alpha_1,\alpha_2} (a_1 a_2 a_1)^{\lambda-1} a_2^{N(\lambda-1)/2-8\rho_2} f(a_1^2, a_2^2) a_1^{8\rho_2} \ldots$$

$$\ldots A_{\epsilon_1,\alpha_1} f(e_1, a_1) e_1^{N(\lambda-1)/2}$$

The integers N are computed based on the quilting.

This yields the relation (IV):

$$f(e_1, a_1) a_3^{-8\rho_2} f(a_2^2, a_3^2)(a_3 a_2 a_3)^{2m} f(e_2, e_1) e_2^{2m} \ldots$$

$$f(e_3, e_2) a_2^{-2m} (a_1 a_2 a_1)^{2m} f(a_1^2, a_2^2) a_1^{8\rho_2} f(a_3, e_3) = 1$$

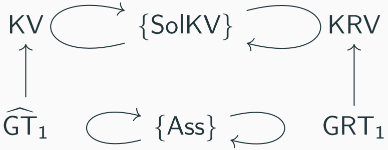
Groups Related to $\widehat{\mathsf{GT}}$: The higher genus case

When you add in the higher genus surfaces several things become more complicated.

- We no longer have the ability to compare with ribbon braids outside of genus 0;
- The classifying spaces $B\mathcal{S}_{g,n}$ are not "nice" for $g \geq 2$;
- There are several subtle homotopy theory problems that aren't yet known for ∞-modular operads.
- But! We expect similar results to the BHR paper in higher genus.

Groups Related to $\widehat{\mathsf{GT}}$: The KV Symmetry Groups

There is another interesting group related to the prounipotent $\widehat{\mathsf{GT}}$: The symmetry groups of the **Kashiware-Vergne** solutions which fit into the following diagram:



Groups Related to $\widehat{\mathsf{GT}}$: The KV Symmetry Groups

Alekseev and Torrossian conjecture that:

$$\mathsf{KRV} \cong \mathsf{GRT}_1 \times \mathbb{Q}$$

and

$$\mathsf{KV} \cong \widehat{\mathsf{GT}} \times \mathbb{Q}.$$

Groups Related to $\widehat{\mathsf{GT}}$: The KV Symmetry Groups

In a recent paper with Dancso and Halacheva we show:

$$\mathsf{KV} \cong \mathsf{Aut}_0(\widehat{\mathsf{wF}})$$

and

$$\mathsf{KRV} \cong \mathsf{Aut}_0(\mathsf{A})$$

where wF and A are **wheeled props with orientation switches** or equivalently **non-connected modular operads**.

Question for the audience: Does a profinite version of this problem make sense?

Thanks!