1. Lecture 1: Graphs and Modular Operads

1.1. **Modular Operads.** We will start today with a very general definition of coloured *modular operads.* For this first lecture we will only discuss discreet, or set-based, coloured modular operads. In the following lectures we will specialize more and more to include examples of space-valued one coloured modular operads.

Definition 1.1. A $\mathfrak{C}$-coloured *cyclic operad* is an algebraic structure consisting of:

- an involutive set of colours $\mathfrak{C}$;
- for each $c_1, \ldots, c_n \in \mathfrak{C}$ a Σ_n-set $\mathbf{P}(c_1, \ldots, c_n)$;
- a family of equivariant, associative and unital composition operations

$$\mathbf{P}(c_1, \ldots, c_n) \times \mathbf{P}(d_1, \ldots, d_m) \longrightarrow \mathbf{P}(c_1, \ldots, \hat{c}_i, \ldots, d_1, \ldots \hat{d}_j, \ldots, d_m),$$

whenever $c_i = d_j$, $(i, j) \in [1, n] \times [1, m]$.

A $\mathfrak{C}$-coloured *modular operad* is a cyclic operad which also has

- a family of equivariant contraction operations

$$\mathbf{P}(c_1, \ldots, c_n) \longrightarrow \mathbf{P}(c_1, \ldots, \hat{c}_i, \ldots, \hat{c}_j, \ldots c_n),$$

whenever $c_i = c_j$, $0 \leq i < j \leq n$.

Moreover, we require all of these operations to satisfy some axioms. **add references**

Remark 1.2. Most examples of coloured modular operads in the literature have been in the setting where the involution on colour sets is trivial. A notable exception is [Pet13], which had a class of examples which were coloured by involutive groupoids, rather than involutive sets. Working with involutive color sets presents distinct homotopical advantages, but it also has the advantage that we can regard wheeled properads as a special case of modular operads.

Example 1.3. The primary example we will discuss in Lecture 3 is a one-coloured modular operad where n-ary operations are surfaces with n boundaries. An example of such an n-ary operation is depicted below by a genus 0 surface with 3 boundaries. The composition operations are given by gluing boundaries. The cyclic structure tells us that we can glue together any two boundaries. Contraction operations tell us we can have "self-gluings" –allowing us to build up surfaces of higher genus.

insert pictures

There are many other well-known cyclic and modular operads in the literature, but we want to emphasise that the definition we've given of cyclic and modular operads also captures some interesting categorical examples one might not naturally think of as cyclic or modular operads.

Exercise 1.4. $*$-autonomous categories are closed symmetric monoidal categories with a global dualizing object so that $(a^\dagger)^\dagger \cong a$. Show that all strict $*$-autonomous categories are examples of cyclic operads. (See [DCH21, Example 2.2]).

1.2. **Graphs.** The goal for today is to introduce you to a category of graphs we call $\mathbf{U}$ and show that there is an equivalence of categories:

$$\mathbf{ModOp} \xrightarrow[\cong]{N} \mathbf{Set}^{\mathbf{U}^{op}}_{Segal}.$$

Our graphs will be *undirected* and are allowed to have "loose ends" – meaning that it is not necessary for both ends (or either end) of an edge to touch a vertex. A typical example is in Figure 1. We will use a combinatorial definition of graphs, *Feynman graphs*, due to Joyal and Kock [JK09]. This model for graphs has the advantage that it is extremely easy to write down and the drawback in that it does not fully capture all of the graphs we need for defining modular operads, but we will return to this issue later in the lecture series.

The edges of our graphs are all comprised of two *distinct* "half edges" which you can picture as a copy of the interval $(0, 1)$ equipped with a chosen orientation. Half-edges are assigned to vertices by a partial function $t : A \to V$. Since not all half-edges will be attached to a vertex and we will write $D \subseteq A$ for those which are in the *domain* of the function t. These half-edges get glued together to form edges by a free involution i which identifies each half-edge with another half-edge with the opposite orientation.

Definition 1.5. A *graph* G consists of:

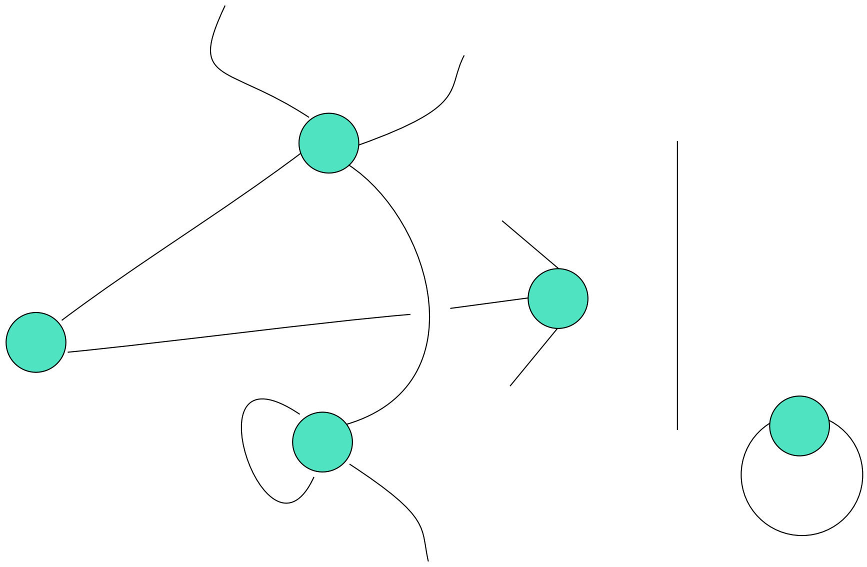


FIGURE 1

- a diagram of finite sets

$$i \circlearrowright A \xleftarrow{s} D \xrightarrow{t} V$$

  where
  - i is a fixed point-free involution and
  - s is a monomorphism.

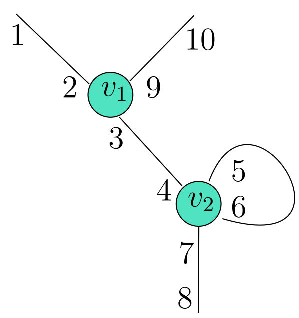


FIGURE 2. $A = \{1, 2, \ldots, 9, 10\}$, $V = \{v_1, v_2\}$, $i(2n) = 2n - 1$

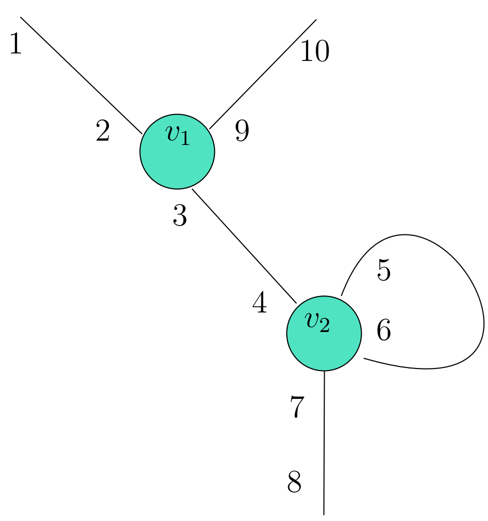


FIGURE 3. The two internal edges: $e_1 = [3, 4]$ and $e_2 = [5, 6]$.

The involution on half-edges $i : a \mapsto a^\dagger$ determines the *edges* of a graph.

- An *edge* is just an i-orbit $[a, a^\dagger]$. We write $E(G) = A/i$ for the set of edges of the graph G.
- An *internal edge* is an edge of the form $[b, b^\dagger]$ where both b and $b^\dagger$ are in D.

Below we have two important examples of graphs: the *exceptional edge* and the graph on the right is the *star* with n loose ends.

Definition 1.6. • The *boundary* of a graph is the set $\partial(G) = A \setminus D$.

- For any vertex $v \in V(G)$ the *neighbourhood* of the vertex consists of the arcs adjacent to the vertex:

$$\mathrm{nb}(v) = t^{-1}(v) \subseteq D.$$

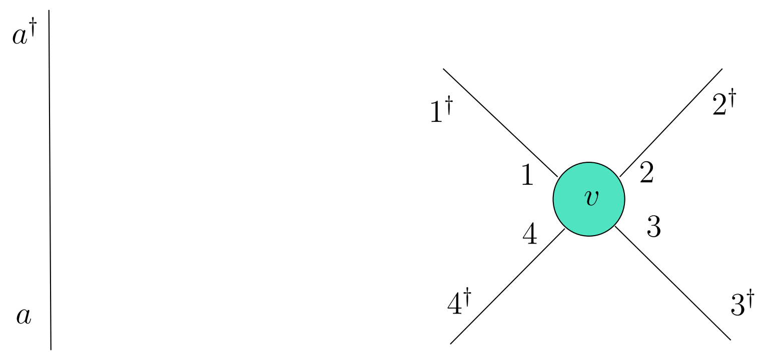
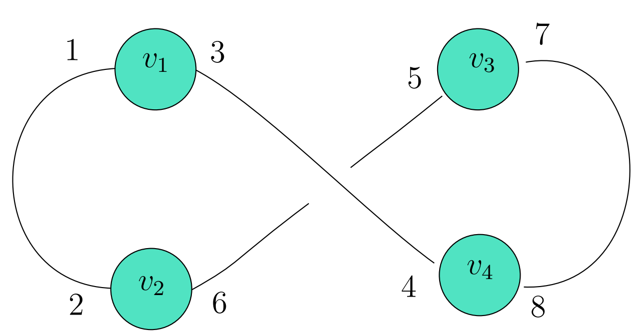


FIGURE 4. The exceptional edge $\updownarrow$ and the 4-star $\star_4$.

A lot of our graphs that have empty boundary such as the loop with n-vertices depicted below (Example 5). There is one graph that we *cannot* describe using our chosen definition of graphs and that is the *nodeless loop*. The nodeless loop would have $A = \{\uparrow, \downarrow\}$ and $D = V = \emptyset$ and can not be distinguished from the *exceptional edge*. We can use the notion of boundary to distinguish the nodeless loop and the exceptional edge: the first has empty boundary and the later has non-empty boundary.

FIGURE 5. A loop with 2 nodes and a nodeless loop.

Exercise 1.7. Draw a graph G with $A = \{1,2,3,4,5,6,7,8\}$, $D = \{1,2,3,4,5,6,7,8\}$, $V = \{v_1, v_2, v_3, v_4\}$ and $i(n) = n - 1$ for $n = 2, 4, 6, 8$.



Definition 1.8.
- Let $\bigstar_G$ be the one-vertex graph with $A = \partial(G) \sqcup \partial(G)^\dagger$ and $D = \partial(G)^\dagger$. Notice that we must have $\partial(\bigstar_G) = A \backslash D = \partial(G)$ and that the neighbourhood of the unique vertex is $D = \partial(G)^\dagger$. In other words, $\bigstar_G = \bigstar_{\partial(G)}$.
- Suppose that v is a vertex of G and let $\mathrm{nb}(v)$ be its neighbourhood in G. We let $\bigstar_v$ denote the graph with $V(G) = \{v\}$, $D = \mathrm{nb}(v)$, and $A = \mathrm{nb}(v) \sqcup \mathrm{nb}(v)^\dagger$. The boundary of $\partial(\bigstar_v) = \mathrm{nb}(v)^\dagger$.

Example 1.9. For the graph G drawn above, write down $\bigstar_G$ and $\bigstar_v$ for each $v \in V(G)$.

1.3. **Morphisms of Graphs.** The graphs we described (except the nodeless loop) will be objects in our graphical category $\mathbf{U}$. Our notion of a graphical morphism will include a notion of 'blowing up' vertices of a graph G with other graphs in a way that reflects iterated compositions in a modular operad. The restrictions we make on graphical morphisms have many advantages: our resulting category will have nice factorisation properties and our category eliminates and morphisms that would translate to 'duplication of variables' in modular operads – following the general theory of operads, which generally can model types of algebras where each variable term appears exactly once in the defining equations.

Graphs are diagrams in **FinSet** in the shape of

$$\mathcal{I} := \quad i \circlearrowright \bullet \xleftarrow{s} \bullet \xrightarrow{t} \bullet$$

where the leftward arrow is sent to a monomorphism and the generating endomorphism is sent to a free involution. Morphisms in our graphical category are, in general, *not* morphisms in the functor category $\mathbf{FinSet}^{\mathcal{I}}$, but an important class of natural transformations in $\mathbf{FinSet}^{\mathcal{I}}$ called *embeddings* are graphical maps.

Definition 1.10 (Natural Transformations of Graphs)**.**

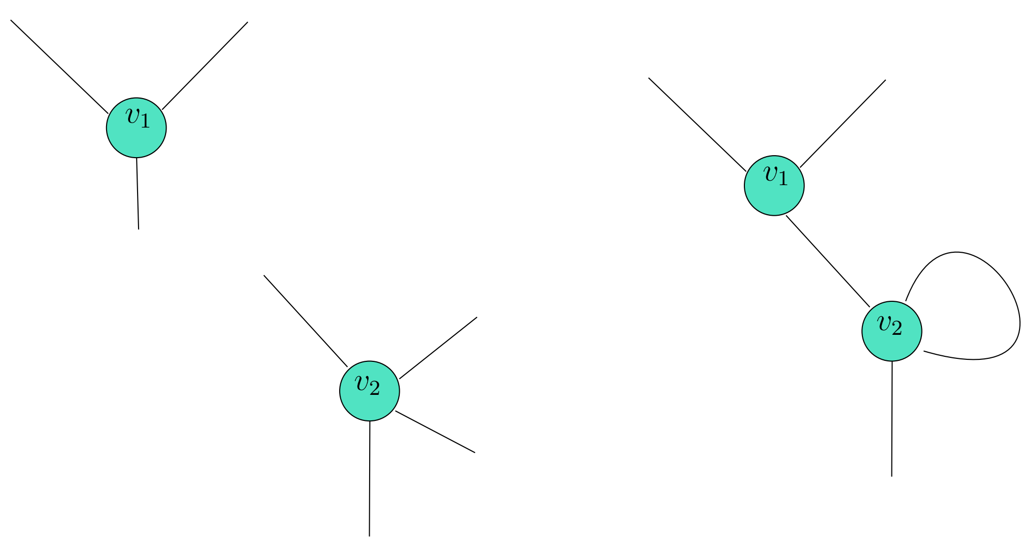
- If G and G' are two graphs, then a natural transformation $G \to G'$ is said to be étale if the right-hand square of:

$$\begin{array}{ccccc} i \circlearrowright A & \xleftarrow{s} & D & \xrightarrow{t} & V \\ \downarrow & & \downarrow & & \downarrow \\ i' \circlearrowright A' & \xleftarrow{s'} & D' & \xrightarrow{t'} & V' \end{array}$$

  is a pullback.
- If G and G' connected graphs, then an étale map is called an *embedding* if $V \to V'$ a monomorphism.

Embeddings capture the idea of a *subgraph*. An important class of embeddings are the is a canonical embeddings $\bigstar_v \hookrightarrow G$ for every $v \in V(G)$:

Example 1.11. Suppose that v is a vertex of G and let $\mathrm{nb}(v)$ be its neighbourhood in G. We let $\bigstar_v$ denote the graph with $V(G) = \{v\}$, $D = \mathrm{nb}(v)$, and $A = \mathrm{nb}(v) \sqcup \mathrm{nb}(v)^\dagger$. The boundary of $\partial(\bigstar_v) = \mathrm{nb}(v)^\dagger$.



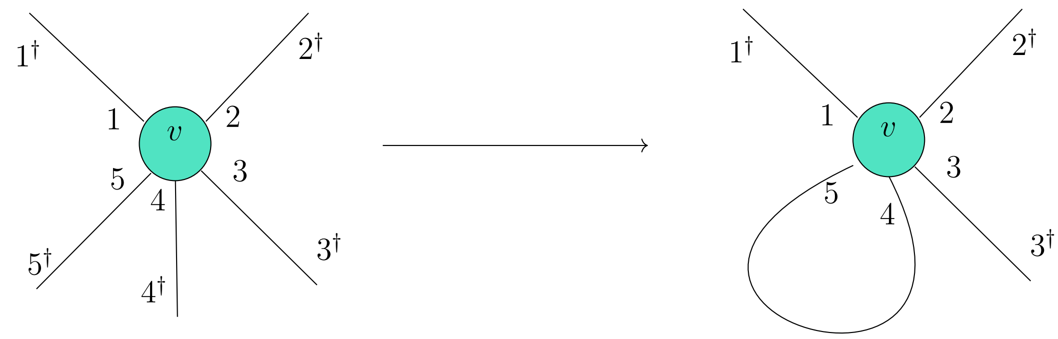
Explicitly:

$$\begin{array}{ccccc} \mathrm{nb}(v) \sqcup \mathrm{nb}(v)^\dagger & \xleftarrow{s} & \mathrm{nb}(v) & \xrightarrow{t} & \{v\} \\ \downarrow & & \downarrow & & \downarrow \\ A & \xleftarrow{s'} & D & \xrightarrow{t} & V. \end{array}$$

The left-hand map in this diagram is just the inclusion $\mathrm{nb}(v) \to D \to A$ on the first component, while the second component (which is forced by compatibility with the involutions) sends $a^\dagger$ to ia.

Embeddings are not necessarily injective on half-edges.

Example 1.12. That is, we consider the graph G with one vertex v, set of arcs $A = \{1, 2, 3, 4, 5, 1^\dagger, 2^\dagger, 3^\dagger\}$ and $D = \{1, 2, 3, 4, 5\}$. The involution $i(n) = n^\dagger$, $n = 1, 2, 3$ and $i(4) = 5$ (and $i(5) = 4$). There is a natural embedding $i_v : \bigstar_5 \hookrightarrow G$ which is not injective on half-edges.



We are now ready to define graphical maps:

Definition 1.13. A graphical map $\varphi : G \to G'$ consists of:

- a map of involutive sets $\varphi_0 : A \to A'$;
- a function $\varphi_1 : V \to \mathrm{Emb}(G')$ satisfying the following conditions:
  - The embeddings $\varphi_1(v)$ do not *overlap* at vertices – no vertex w in G' is contained in two subgraphs $\varphi_1(v)$ and $\varphi_1(v')$;
  - For each v, we have a (necessarily unique) bijection making the diagram:

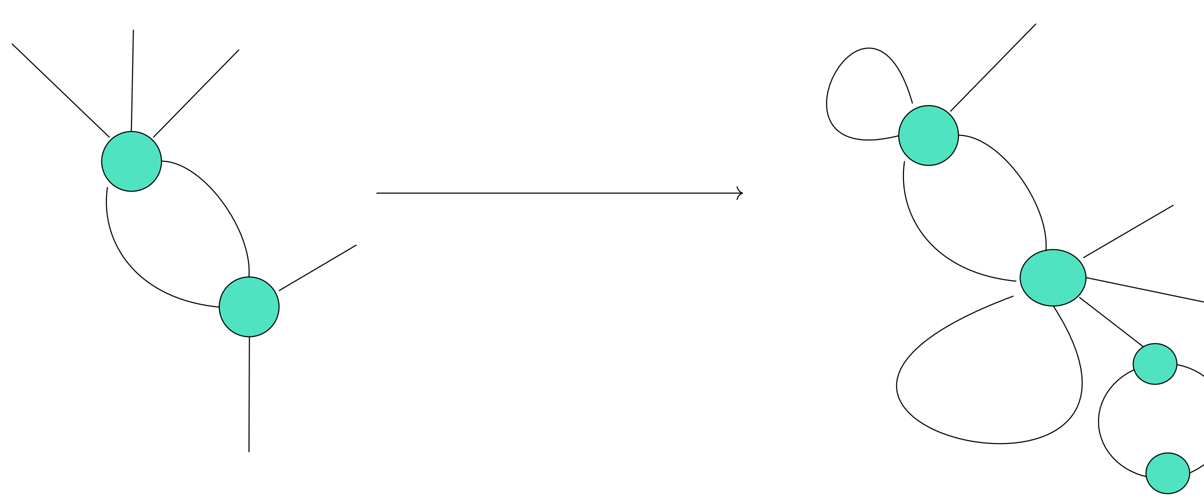
$$\begin{array}{ccc} \mathrm{nb}(v) & \xrightarrow{\ i\ } & A \\ \cong\downarrow & & \downarrow\varphi_0 \\ \partial(\varphi_1(v)) & \longrightarrow & A' \end{array}$$

    commute, where the top map i is the restriction of the involution on A.
  - If $\partial(G) = \emptyset$, then there exists a v in V so that $\varphi_1(v) \neq \updownarrow$.

The first two conditions basically say that a map $\varphi : G \to G'$ is obtained by "blowing up" the vertices of G into a graph

$$H_v \xhookrightarrow{\varphi_1(v)} G' \ .$$

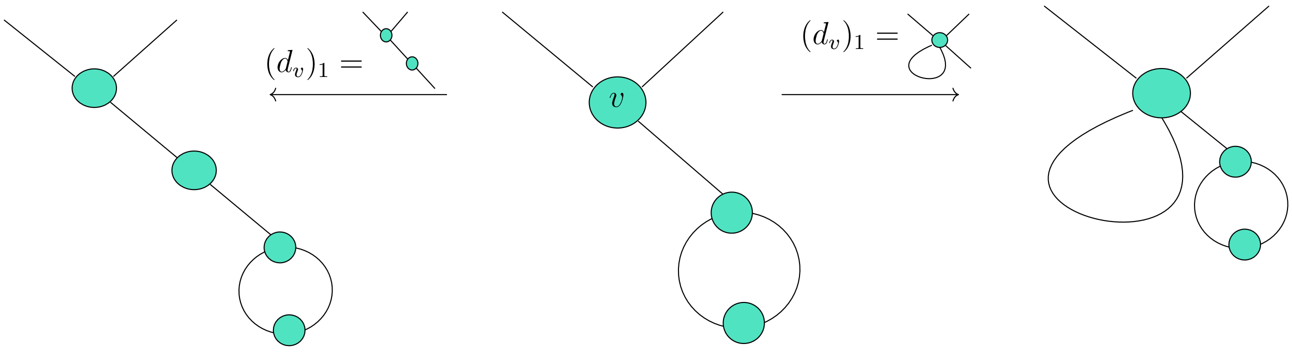
The third condition is about avoiding the collapse into a *nodeless loop*. In the picture below we have circled the the embedded subgraphs $\varphi_1(v), \varphi_1(v')$ of G'. The notion of "blowing up" a vertex can be made precise using the language of *graph substitution* which is described for Feynman graphs in Construction 1.18 [HRY20a].



While the definition of graphical map is complicated, all graphical maps can be described (up to isomorphism) as the composite of three elementary classes of graphical maps: inner coface maps, outer coface maps and codegeneracies (Theorem 2.7 [HRY20a]).

Definition 1.14. An *inner coface map* $d_v : G \to G'$ is a graphical map defined by blowing-up a single vertex v in G by a graph $(d_v)_1$ which has precisely *one* internal edge.

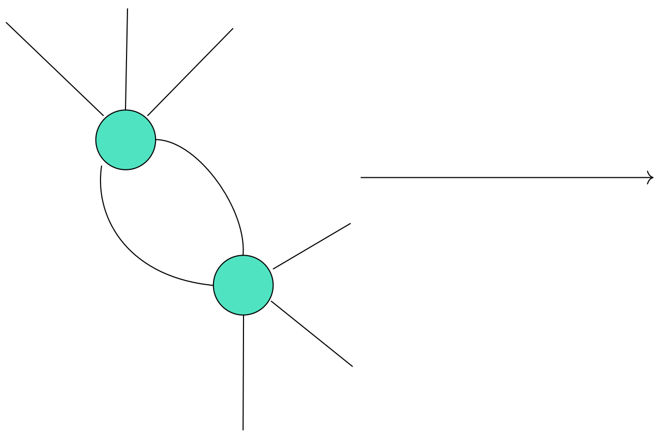
Example 1.15.



Definition 1.16. An *outer coface map* is either:

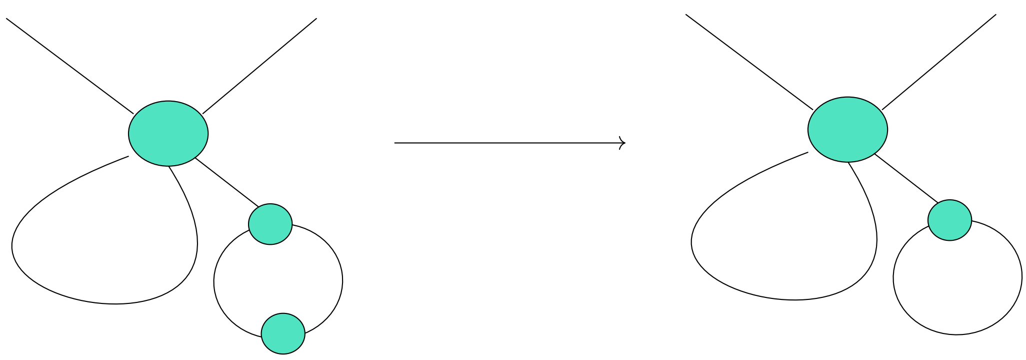
- an *embedding* $d_e : G \to G'$ in which G' has precisely *one* more internal edge than G or
- an *embedding* $\updownarrow \to \bigstar_n$.

Example 1.17.



Definition 1.18. A *codegeneracy map* $s_v : G \to G'$ is a graphical map defined by "blowing-up" a vertex v in G by $\updownarrow$.

Example 1.19.



Definition 1.20. The graphical category $\mathbf{U}$ is the category whose objects are connected Feynman graphs (Definition 1.5). The morphisms are the graphical maps from Definition 1.13.

For the remainder of this first lecture we will describe how this graphical category $\mathbf{U}$ relates to modular operads. We will show that every object of $\mathbf{U}$ generates a modular operad and there is a faithful functor

$$J : \mathbf{U} \longrightarrow \mathbf{ModOp}$$

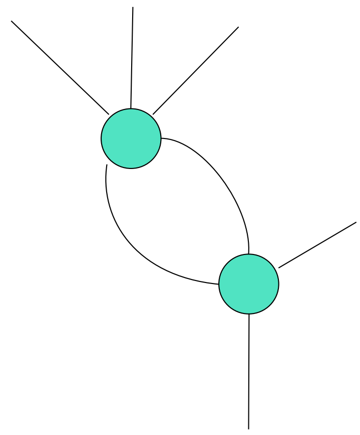
which is injective on isomorphism classes of objects. We will also define a category of graphical sets: set-valued pre-sheaves of $\mathbf{U}$. We will show that graphical sets which satisfy a strict Segal condition are necessarily the nerve of a modular operad. In the next lecture we will show how to weaken the Segal condition on space-valued graphical pre-sheaves to define a model for ∞-modular operads.

The modular operad $\langle G \rangle$ generated by a graph G is the free modular operad whose:

- set of colours is the set of half-edges A;
- a collection of Σ_n-sets is $E(a_1, \ldots, a_n) = \begin{cases} \{v\} \text{ if } (a_1, \ldots, a_n) = \partial(\star_v) \\ \emptyset \text{ otherwise.} \end{cases}$

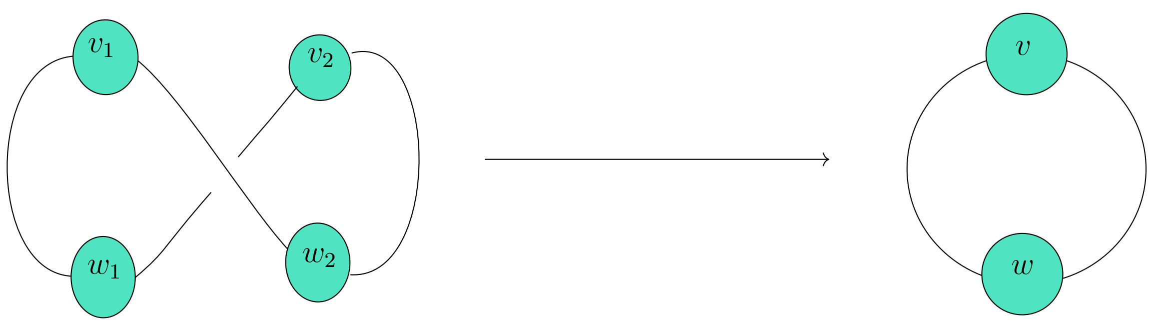
- $\langle G \rangle = F(E)$

Example 1.21. The individual corolla create your building blocks for the modular operad – the graph gives gluing instructions



Proposition 1.22 (Proposition 2.25 [HRY20b])**.** *The assignment $G \mapsto \langle G \rangle$ defines a faithful functor* $\mathbf{U} \to \mathbf{ModOp}$ *which is injective on isomorphism classes of objects.*

We note that the functor is not full. To see this one can consider the graphs G and G' below. There is a map of modular operads from $\langle G \rangle$ to $\langle G' \rangle$ which sends each v_i to v and each w_j to w but there is no graphical map $G \to G'$ which has this behaviour.



1.4. **Graphical Sets.** At this point, we have defined a faithful functor $\mathbf{U} \to \mathbf{ModOp}$. There is an associated singular functor, or *nerve functor*, which goes from the category of modular operads $\mathbf{ModOp}$ to the category of set-valued $\mathbf{U}$-presheaves.

Definition 1.23. The category of graphical sets is the category of set-valued $\mathbf{U}$ presheaves. Objects are functors $X : \mathbf{U}^{op} \to \mathbf{Set}$ and the morphisms are natural transformations.

- We write X_G for the evaluation of X at a graph $G \in \mathbf{U}$.
- For every morphism $\varphi : G \to G'$, there's map $\varphi^* : X_{G'} \to X_G$.
- For any graph G in $\mathbf{U}$ the representable presheaf

$$\mathbf{U}[G] := \mathbf{U}(-; G)$$

  given by

$$\mathbf{U}[G]_H := \mathbf{U}(H, G)$$

  for all graphs H.

We think of an element $x \in X_G$ as a "decoration" of shape G. The Yoneda Lemma tells us that there is a bijection identifying every $x \in X_G$ with an element $x \in \mathbf{Set}^{\mathbf{U}^{op}}(\mathbf{U}[G], X)$.

If G is a graph with at least two vertices, we note that, for any internal edge between two vertices v and w in G we have a diagram of embeddings

$$\bigstar_v \longleftarrow \updownarrow \longrightarrow \bigstar_w$$

in $\mathbf{U}$. We define

$$X_G^1 = \lim_{\bigstar_v \leftarrow \updownarrow \rightarrow \bigstar_w} \left(\begin{array}{ccc} X_{\bigstar_v} & & X_{\bigstar_w} \\ & \searrow \quad \swarrow & \\ & X_{\updownarrow} & \end{array} \right).$$

The embeddings $\bigstar_v \hookrightarrow G$ induce a Segal map

$$X_G \longrightarrow X_G^1 \subseteq \prod_{v \in V(G)} X_{\bigstar_v}$$

which factors through this limit.

Exercise 1.24. In the case when $X_{\updownarrow} = *$, show that $X^1_G = \prod_{v \in V(G)} X_{\bigstar_v}$.

Definition 1.25. A graphical set $X \in \mathbf{Set}^{\mathbf{U}^{op}}$ is strictly *Segal* if the Segal map is a bijection for each G in $\mathbf{U}$.

The intuition of this map says that "decorations of the graph G" are determined by the "decorations at each vertex".

Definition 1.26. The *nerve* functor

$$N : \mathbf{ModOp} \longrightarrow \mathbf{Set}^{\mathbf{U}^{op}}$$

is given by

$$N\mathbf{P}_G = \mathbf{ModOp}(\langle G \rangle, \mathbf{P})$$

for any modular operad $\mathbf{P}$ and any graph $G \in \mathbf{U}$.

We can think of $N\mathbf{P}_G$ as the set of $\mathbf{P}$ decorations of the graph G:

- $N\mathbf{P}_{\updownarrow} = \mathbf{ModOp}(\langle \updownarrow \rangle, \mathbf{P}) = \mathfrak{C}$;
- $N\mathbf{P}_{\star_n} = \mathbf{ModOp}(\langle \star_n \rangle, \mathbf{P}) = \mathbf{P}(c_1, \dots c_n)$. **Insert picture**

Theorem 1.27. [HRY20b, Theorem 3.6] *The nerve functor*

$$N : \mathbf{ModOp} \longrightarrow \mathbf{Set}^{\mathbf{U}^{op}}$$

is fully faithful. Moreover, the following statements are equivalent for $X \in \mathbf{Set}^{\mathbf{U}^{op}}$.

(1) There exists a modular operad $\mathbf{P}$ and an isomorphism $X \cong N\mathbf{P}$.
(2) X satisfies the strict Segal condition.

Remark 1.28. Sophie Raynor had a similar theorem in her PhD thesis using a slightly larger graphical category and the "Webber nerve."

Exercise 1.29. Given a graph $G \in \mathbf{U}$, we now have two ways to assign an object in $\mathbf{Set}^{\mathbf{U}^{op}}$ to G:

- the representable presheaf $\mathbf{U}[G]$,
- taking the nerve of the modular operad $\langle G \rangle$, $N\langle G \rangle$.

The representable $\mathbf{U}[G]$ is a sub-object of $N\langle G \rangle$ (since $J : \mathbf{U} \to \mathbf{ModOp}$ is faithful) but they nearly never coincide.

- Let G be the loop with one node and show $\mathbf{U}[G] \subset N\langle G \rangle$.
- Show that we have $\mathbf{U}[\star_0] = N\langle \star_0 \rangle$.

1.5. **Further Directions.** Earlier we defined the notion of (inner and outer) coface maps of $\mathbf{U}$. Given a coface map δ with codomain G, one can define the **horn** $\Lambda^\delta[G]$ which is a sub-object of the representable object $\mathbf{U}[G]$. A *strict inner Kan* graphical set is a presheaf $X \in \mathbf{Sets}^{\mathbf{U}^{op}}$ such that every diagram

$$\begin{array}{ccc} \Lambda^\delta[G] & \longrightarrow & X \\ \downarrow & \nearrow & \\ \mathbf{U}[G] & & \end{array}$$

with δ an inner coface map admits a unique filler. Michelle Strumila shows in her PhD thesis that :

Theorem 1.30 (Strumila)**.** *The nerve functor*

$$N : \mathbf{ModOp} \longrightarrow \mathbf{Set}^{\mathbf{U}^{op}}$$

is fully faithful. Moreover, the following statements are equivalent for $X \in \mathbf{Set}^{\mathbf{U}^{op}}$.

(1) There exists a modular operad $\mathbf{P}$ and an isomorphism $X \cong N\mathbf{P}$.
(2) X satisfies the strict Segal condition.
(3) X is strict inner Kan.

If one relaxes the inner Kan condition you arrive at a model for *quasi* or ∞-*modular operads*. Following the example of dendroidal sets, one could find a model category structure in which the weak inner Kan graphical sets are the fibrant objects. Because the graphical categories for cyclic operads, wheeled properads, etc can all be derived from $\mathbf{U}$ this would simultaneously create models for many flavours of ∞-"operads".

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