1. Lecture 2: A weak Segal model for ∞-modular operads

Yesterday we introduced modular operads and defined a category of graphs $\mathbf{U}$ which model modular operads in the sense that there is an equivalence of categories:

$$\mathbf{ModOp} \xrightarrow[\cong]{N} \mathbf{Set}^{\mathbf{U}^{op}}_{Segal}.$$

In the last talk we used a relatively general definition of an $\mathfrak{C}$-coloured modular operad, but from now on we are going to specialise to *one-coloured* modular operads. Just to refresh our memory a *modular operad* consists of a collection $\mathbf{P} = \{\mathbf{P}(n)\}$ where:

- Each $\mathbf{P}(n)$ has a Σ_n-action;
- A family of equivariant associative compositions

$$\mathbf{P}(n) \times \mathbf{P}(m) \xrightarrow{\circ_{ij}} \mathbf{P}(n+m-2);$$

- A family of equivariant contraction operations

$$\mathbf{P}(n) \xrightarrow{\xi_{ij}} \mathbf{P}(n-2)$$

which satisfy some axioms.

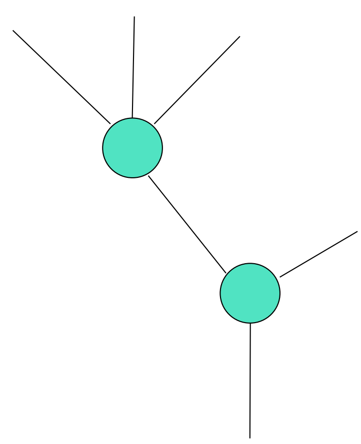
What the theorem from yesterday tells us is that, given a (one-coloured, discrete) modular operad $\mathbf{P}$, we can construct a set-valued presheaf $N\mathbf{P} \in \mathbf{Set}^{\mathbf{U}^{op}}_{Segal}$ where

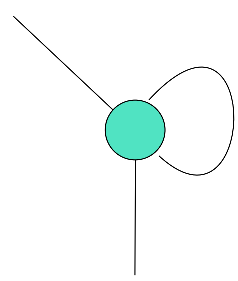
$$N\mathbf{P}_{\bigstar_n} = \mathbf{P}(n)$$

and the Segal maps

$$N\mathbf{P}_G \longrightarrow \prod_{v \in V(G)} N\mathbf{P}_{\bigstar_v}$$

are precisely the modular operad compositions and contractions. **Insert picture**





In the next lecture we want to consider a situation when we have a one-coloured (groupoid or space-valued modular) operad $\mathbf{P} = \{\mathbf{P}(n)\}$ which we will *profinitely complete* entry-wise. Profinite completion does not commute with products of spaces,

$$\widehat{X \times Y} \neq \widehat{X} \times \widehat{Y},$$

and so $\widehat{\mathbf{P}} = \{\widehat{\mathbf{P}(n)}\}$ will no longer form a (modular) operad we will, however, show that under very special circumstances the corresponding Segal maps:

$$N\mathbf{P}_G \longrightarrow \prod_{v \in V(G)} N\mathbf{P}(\bigstar_v)$$

are weak homotopy equivalences of spaces.

Our goals today:

- Introduce space-valued presheaves $\mathbf{sSet}^{\mathbf{U}^{op}}$;
- Describe the Segal condition on graphical spaces;
- Profinite completion of (modular) operads;
- Discuss how variations on the graphical category $\mathbf{U}$ can give genus graded ∞-modular operads, cyclic operads, etc.
- Open Problems

1.1. **Graphical Spaces.** Let **sSet** be category of simplicial sets which has the Kan-Quillen model structure. I will very often abuse terminology and refer to simplicial sets as “spaces”.

Definition 1.1. The category of *graphical spaces* is the category of space-valued **U**-presheaves by $\mathbf{sSet}^{\mathbf{U}^{op}}$.

Let’s recall a bit of notation from the previous lecture:

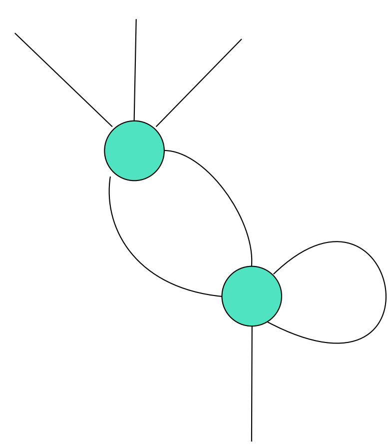
- For any $X \in \mathbf{sSet}^{\mathbf{U}^{op}}$ we write X_G for the evaluation of X at a graph $G \in \mathbf{U}$.
- We write $\mathbf{U}[G] \in \mathbf{Set}^{\mathbf{U}^{op}}$ for the representable presheaf at a graph G:
$$\mathbf{U}[G]_H = \mathbf{U}(H, G)$$
where H ranges over all graphs in $\mathbf{U}$.
- We consider $\mathbf{U}[G]$ as an object in $\mathbf{sSet}^{\mathbf{U}^{op}}$ via the inclusion $\mathbf{Set}^{\mathbf{U}^{op}} \hookrightarrow \mathbf{sSet}^{\mathbf{U}^{op}}$.

Exercise 1.2. The Yoneda Lemma says that a map $x : \mathbf{U}[G] \to X$ in $\mathbf{Set}^{\mathbf{U}^{op}}$ is equivalent to an element $x \in X_G$. Show that every $X \in \mathbf{Set}^{\mathbf{U}^{op}}$ is, up to isomorphism, a colimit of representables
$$X \cong \operatorname{colim} \mathbf{U}[G]$$
where the colimit is indexed by the maps $\mathbf{U}[G] \to X$.

To understand the Segal core it can be useful to revisit the definition of a graph. A graph G (which has at least one vertex) is made up of the stars at each vertex and “glued” together along each internal edge. Indeed, for G a connected graph with $V \neq \emptyset$ we can present G as a coequalizer in $\mathbf{FinSet}^{\mathcal{I}}$.

$$\coprod_{e \in \mathrm{iE}} \updownarrow \rightrightarrows \coprod_{v \in V} \bigstar_v \longrightarrow G$$

by choosing an orientation for each internal edge – this gives us a decomposition of the graph. For example:



Not this is not a coequalizer in $\mathbf{U}$ as these objects don’t exist in $\mathbf{U}$ (they are not connected).

Definition 1.3. The *Segal core* of a graph G is the coequalizer in $\mathbf{Set}^{\mathbf{U}^{op}}$:

$$\coprod_{e \in \mathrm{iE}} \mathbf{U}[\updownarrow] \rightrightarrows \coprod_{v \in V} \mathbf{U}[\bigstar_v] \longrightarrow \mathbf{Sc}[G].$$

The Segal core comes with a map $\mathbf{Sc}[G] \to \mathbf{U}[G]$ induced by the embeddings $\bigstar_v \to G$. In the case that $G = \updownarrow$ we declare the map $\mathbf{Sc}[G] \to \mathbf{U}[G]$ to be the identity map on $\mathbf{U}[G]$.

We note that the Segal core definition is precisely the colimit so that

$$\mathbf{Set}^{\mathbf{U}^{op}}(\mathbf{Sc}[G], X) = X_G^1 = \lim_{\bigstar_v \leftarrow \updownarrow \rightarrow \bigstar_w} \begin{pmatrix} X_{\bigstar_v} & & X_{\bigstar_w} \\ & \searrow \quad \swarrow & \\ & X_{\updownarrow} & \end{pmatrix}.$$

In the case that
$$X_{\updownarrow} = \mathbf{Set}^{\mathbf{U}^{op}}(\mathbf{U}[\updownarrow], X) = *$$
then we can identify
$$\mathbf{Set}^{\mathbf{U}^{op}}(\mathbf{Sc}[G], X) = \prod_{v \in V(G)} X_{\bigstar_v}.$$

Remark 1.4. In the case $X_{\updownarrow} = *$, the Segal map is given by:

$$X_G = \mathbf{Set}^{\mathbf{U}^{op}}(\mathbf{U}[G], X) \longrightarrow \mathbf{Set}^{\mathbf{U}^{op}}(\mathbf{Sc}[G], X) = \prod_{v \in V(G)} X_{\bigstar_v}.$$

The goal is to relax the Segal map

$$X_G = \mathbf{sSet}^{\mathbf{U}^{op}}(\mathbf{U}[G], X) \longrightarrow \mathbf{sSet}^{\mathbf{U}^{op}}(\mathbf{Sc}[G], X) = \prod_{v \in V(G)} X_{\bigstar_v}$$

from a bijection of sets to a weak equivalence. A good definition is as follows:

Definition 1.5. A space-valued presheaf $X \in \mathbf{sSet}^{\mathbf{U}^{op}}$ is *Segal* if:

- $X_{\updownarrow} = *$;
- for all $G \in \mathbf{U}$, the Segal map

$$\mathrm{map}^h(\mathbf{U}[G], X) \longrightarrow \mathrm{map}^h(\mathbf{Sc}[G], X)$$

  is a weak equivalence.

Here we write $\mathrm{map}^h(X, Y)$ for the derived mapping space. This is well-defined as long as we can equip the category $\mathbf{sSet}^{\mathbf{U}^{op}}$ with a class of weak equivalences. The category $\mathbf{sSet}^{\mathbf{U}^{op}}$ admits several model category structures including the *projective model structure* and a *Reedy model structure* we will discuss shortly. Both of these model structures have weak equivalences defined "entry-wise", that is $f : X \to Y$ is a weak equivlences if $f : X_G \to Y_G$ is a weak equivalence of simplicial sets for every $G \in \mathbf{U}$.

Remark 1.6. The assumption that $X_{\updownarrow} = *$ is, technically, not required for this definition but is required in a theorem we will state in a moment. It means that we are going to be restricting to modelling one-coloured ∞-modular operads. This is a perfectly fine definition, but, in practice, working with derived mapping spaces is difficult. This simplifies significantly of we use a Reedy model structure on the category $\mathbf{sSet}^{\mathbf{U}^{op}}$.

1.2. **Generalised Reedy Categories.**

Definition 1.7. A dualizable generalized Reedy structure on a small category $\mathbb{R}$ consists of two wide subcategories $\mathbb{R}^+$ and $\mathbb{R}^-$ together with a degree function $\mathrm{ob}(\mathbb{R}) \to \mathbb{N}$ satisfying:

1. non-invertible morphisms in $\mathbb{R}^+$ (respectively $\mathbb{R}^-$) raise (respectively lower degree). Isomorphisms preserve degree.
2. $\mathbb{R}^+ \bigcap \mathbb{R}^- = \mathrm{Iso}(\mathbb{R})$
3. Every morphism f factors as $f = gh$ such that $g \in \mathbb{R}^+$ and $h \in \mathbb{R}^-$. Moreover, this factorisation is unique up to isomorphism.
4. If $\theta f = f$ for any isoorphism θ and $f \in \mathbb{R}^-$ then θ is an identity.
5. If $f\theta = f$ for any isoorphism θ and $f \in \mathbb{R}^+$ then θ is an identity.

The subcategory $\mathbb{R}^+$ is commonly called the 'direct category' and $\mathbb{R}^-$ the 'inverse category.' A category $\mathbb{R}$ that satisfies axioms $(1)-(4)$ is a generalised Reedy category. If, in addition, $\mathbb{R}$ satisfies axiom (5) then $\mathbb{R}$ is said to be dualizable, which implies that $\mathbb{R}^{op}$ is also a generalised Reedy category.

Example 1.8. The simplicial category Δ is a Reedy category with every element of $\mathrm{Iso}(\Delta)$ is an identity.

Example 1.9. Other examples of generalised Reedy categories include the **dendroidal category** Ω, finite sets, pointed finite sets, and the cyclic category Λ.

The main idea of Reedy categories is that we can think about lifting morphisms from $\mathbb{R}$ to the diagram category $\mathcal{M}^{\mathbb{R}}$ by induction on the degree of our objects. To formalise this idea we introduce the notion of latching and matching objects. For any object $r \in \mathbb{R}$, the category $\mathbb{R}^+(r)$ is the full subcategory of $\mathbb{R}^+ \downarrow r$ consisting of those maps with target r which are not invertible. Similarly, the category $\mathbb{R}^-(r)$ is the full subcategory of $r \downarrow \mathbb{R}^-$ consisting of the maps $\alpha : r \to s$ which are not invertible.

Definition 1.10. Let X be a diagram in $\mathcal{M}^{\mathbb{R}}$

- The latching object $L_r X = \mathrm{colim}_{\mathbb{R}^+(r)} X$;
- The matching object $M_r X = \lim_{\mathbb{R}^-(r)} X$.

If $\mathcal{M}$ is a cofibrantly generated model category. We say that a morphism $f : X \to Y$ in $\mathcal{M}^{\mathbb{R}}$ is:

- a Reedy cofibration if $X_r \bigcup_{L_r X} L_r Y \to Y_r$ is a cofibration in $\mathcal{M}^{\mathrm{Aut}(r)}$ for all $r \in \mathbb{R}$;

- a Reedy weak equivalence if $X_r \to Y_r$ in $\mathcal{M}^{\mathrm{Aut}(r)}$ for all $r \in \mathbb{R}$;
- a Reedy fibration if $X_r \to M_r X \times_{M_r Y} Y_r$ in $\mathcal{M}^{\mathrm{Aut}(r)}$ for all $r \in \mathbb{R}$.

Theorem 1.11. [BM11] *If $\mathbb{R}$ is a dualizable generalized Reedy category and $\mathcal{E}$ is a nice enough model category, then $\mathcal{E}^{\mathbb{R}^{op}}$ admits a cofibrantly generated model category structure with level-wise weak equivalences.*

Define the degree of a graph G to be $deg(G) = |V| + |iE|$. Then the degree function $deg : \mathrm{ob}(\mathbf{U}) \to \mathbb{N}$.

Theorem 1.12. [HRY20, Theorem 2.22] *The graphical category $\mathbf{U}$ is a (dualizable) generalised Reedy category. The wide subcategory $\mathbf{U}^-$ is generated by the codegeneracy maps and the wide subcategory $\mathbf{U}^+$ is generated by the inner and outer coface maps.*

Examples showing that these maps raise or lower degree?

Corollary 1.13. *The diagram category $\mathbf{sSet}^{\mathbf{U}^{op}}$ has a model category structure with the Reedy fibrations, Reedy cofibrations, and entry-wise weak equivalences.*

Exercise 1.14 (Hard-ish)**.** In Proposition 3.5 of [HRY20] we show that Segal cores are cofibrant in the Reedy model structure on $\mathbf{sSet}^{\mathbf{U}^{op}}$. Give an example of a graph G in which the Segal core of G fails to be cofibrant in the projective model structure.

The advantage of a Reedy model structure on $\mathbf{sSet}^{\mathbf{U}^{op}}$ is that homotopy limits of Reedy fibrant diagrams are just limits. Revisiting our definition we now have:

Definition 1.15. A space-valued presheaf $X \in \mathbf{sSet}^{\mathbf{U}^{op}}$ is *Segal* if:

- $X_{\updownarrow} = *$;
- X is Reedy fibrant;
- for all $G \in \mathbf{U}$, the Segal map
$$X_G = \mathrm{map}^h(\mathbf{U}[G], X) \longrightarrow \mathrm{map}^h(\mathbf{Sc}[G], X) = \prod_{v \in V(G)} X_{\bigstar_v}$$
is a weak equivalence.

Suppose that $\mathbf{P}$ is a one-coloured modular operad in $\mathbf{sSet}$. If $X = N\mathbf{P}$ is the nerve of $\mathbf{P}$, then $X_{\updownarrow}$ is a point and the Segal map
$$X_G \to \textstyle\prod_{v \in V} X_{\star_v}$$
is an isomorphism for every G. Thus every one-coloured modular operad gives rise to a graphical Segal space. Note, however, that Reedy fibrancy requires some additional assumptions on the modular operad $\mathbf{P}$.

We conclude this description by pointing out that there is a classification of graphical Segal spaces as the fibrant objects in a localisation of the Reedy model category structure on graphical spaces.

Theorem 1.16. *The category $\mathbf{sSet}^{\mathbf{U}^{op}}$ admits a cofibrantly generated model structure whose fibrant objects are the Segal modular operads.*

Remark 1.17. While we have discussed space-valued presheaves everything we've said about weakly Segal modular operads still makes sense for presheaves in a nice enough cartesian monoidal model category $\mathbf{C}$ such as groupoids or categories. You can also relax the requirement that $X_{\updownarrow} = *$ and only require that $X_{\updownarrow}$ be a contractible space.

1.3. profinite completion of operads.

Definition 1.18. For a group G, the **profinite completion** of G is the limit
$$\widehat{G} = \lim G/N$$
where N runs through all normal subgroups of G of finite index.

Example 1.19. The profinite completion of the integers $\widehat{\mathbb{Z}} = \lim \mathbb{Z}/\mathbb{Z}_n$ where the limit is taken over the maps $\mathbb{Z}/\mathbb{Z}_n \to \mathbb{Z}/\mathbb{Z}_m$ for $m \mid n$.

We have seen already in this workshop many examples of operads in groupoids : such as the operad CoB and PaB mentioned in previous talks. Horel shows in [Hor17] that profinite completion extends to groupoids (with finite set of objects) .

As in Damien's talks, where he discussed the prounipotent completion of operads in groupoids, one might be interested in the profinite completion of a (modular) operad $\mathbf{P} = \{\mathbf{P}(n)\}$ in groupoids. As with prounipotent completion, we want to do this entrywise: $\widehat{\mathbf{P}} = \{\widehat{\mathbf{P}(n)}\}$. If our groupoids are nice enough, profinite completion respects products.

Proposition 1.20 (Horel 17)**.** *Let C and D be two groupoids (with finite sets of objects). The map*

$$\widehat{C \times D} \to \widehat{C} \times \widehat{D}$$

induced by the two projections $\widehat{C \times D} \to \widehat{C}$ and $\widehat{C \times D} \to \widehat{D}$ is an isomorphism.

It follows that

$$\widehat{\mathbf{P}} = \{\widehat{\mathbf{P}(n)}\}$$

is a (modular) operad in profinite groupoids. If we put this together with our nerve theorem from yesterday, we know that we can think about $N\widehat{\mathbf{P}}$ as a presheaf $\mathbf{Gr}^{\mathbf{U}^{op}}$ satisfying a strict Segal condition.

$$N\widehat{\mathbf{P}}_G \to \prod_{v \in V} N\widehat{\mathbf{P}}_{\star_v}$$

is an isomorphism of groupoids for every graph G.

To every groupoid we can associate a space by taking the classifying space functor

$$\mathbf{Gr} \xrightarrow{B} \mathbf{sSet}.$$

This functor is symmetric monoidal so that

$$B\mathbf{P} = \{B\mathbf{P}(n)\}$$

is a (modular) operad in spaces. Profinite completion of spaces is not so well behaved however.

Definition 1.21. A discrete group G is said to be *good* if for any finite abelian group M equipped with a G-action the map $G \to \widehat{G}$ induces an isomorphism

$$H^i(\widehat{G}, M) \to H^i(G, M).$$

Proposition 1.22 (BHR)**.** *Let X and Y be two connected spaces whose homotopy groups are good. Then the map*

$$\widehat{X \times Y} \longrightarrow \widehat{X} \times \widehat{Y}$$

is a weak equivalence of profinite spaces.

$\Rightarrow$ In the case that every space of $B\mathbf{P} = \{B\mathbf{P}(n)\}$ satisfies the conditions of the proposition then

$$N\widehat{B\mathbf{P}}_G \to \prod_{v \in V} N\widehat{B\mathbf{P}}_{\star_v}$$

is a weak equivalence for all graphs and $N\widehat{B\mathbf{P}}$ is a Segal (modular) operad.

1.4. **Variations on the graphical category U.** The original definition of modular operad had an additional 'genus' grading and the underlying objects satisfied a stability condition. There is a genus graded version of the graphical category $\mathbf{U}$ and a corresponding stable version of Segal modular operads.

Definition 1.23. Let G be a graph:

(1) A genus function for G is a function $g : V(G) \to \mathbb{N}$.
(2) The total genus of a pair $(G, g : V \to \mathbb{N})$ is given by:

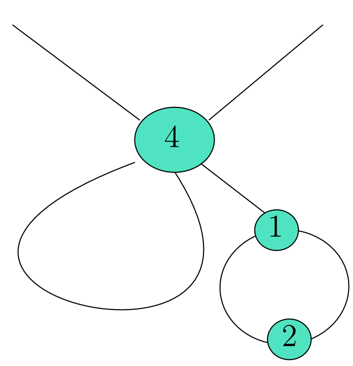
$$g(G) = \beta_1(G) + \Sigma_{v \in V} g(v)$$

where $\beta_1(G)$ is the Betti number of G.
(3) A pair (G, g) is called **stable** if G is connected and for every vertex v:

$$2g(v) + |\operatorname{nb}(v)| - 2 > 0.$$

The exceptional edge admits only one genus function g, and $G(\updownarrow) = \beta_1(\updownarrow) = 0$. This graph trivially satisfies the stability condition.

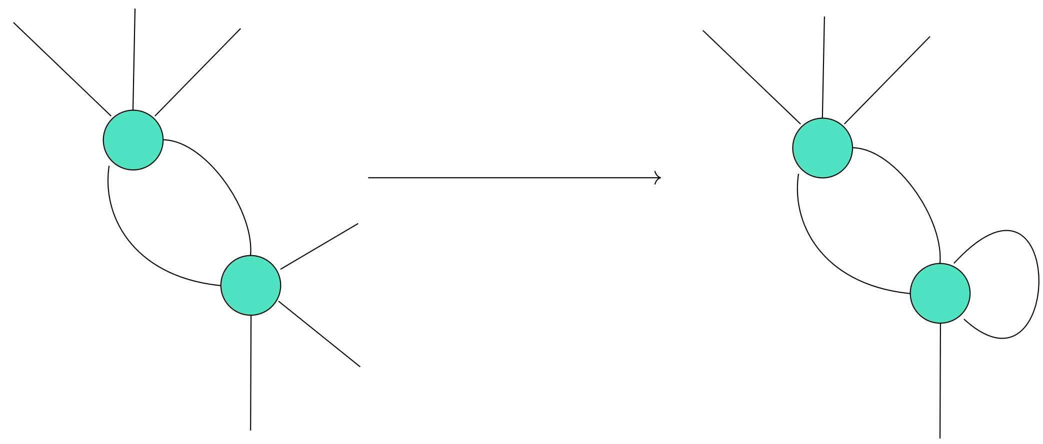
Example 1.24.



Remark 1.25. If $f : H \to G$ is an embedding, then we can define the genus of f

$$g(f) = \beta_1(H) + \Sigma_{v \in V(H)} g(f(v)).$$

Example 1.26.



Definition 1.27. The stable graphical category $\mathbf{U}_{st}$ has:

- Objects: stable graphs (G, g)
- Morphisms: $(G, g) \to (G', g')$ are graphical maps $\varphi : G \to G'$ which make the diagram:

$$\begin{array}{ccc} V(G) & \xrightarrow{\varphi_1} & \mathrm{Emb}(G') \\ & {\scriptstyle g} \searrow \quad \swarrow {\scriptstyle g'} & \\ & \mathbb{N} & \end{array}$$

  commute.

Theorem 1.28. $\mathbf{U}_{st}$ *is a generalized Reedy category.*

Definition 1.29. There is model structure on stable graphical spaces in which $X \in \mathbf{sSet}^{\mathbf{U}_{st}^{op}}$ is fibrant if:

- $X_{\updownarrow} = *$;
- X is Reedy fibrant;
- for all $G \in \mathbf{U}$, the Segal map

$$X_{(G,g)} = \mathrm{map}^h(\mathbf{U}[(G,g)], X) \longrightarrow \mathrm{map}^h(\mathbf{Sc}[(G,g)], X)$$

  is a weak equivalence.

1.5. **Cyclic Operads.** Various subcategories of $\mathbf{U}$ correspond to cyclic operads.

Definition 1.30. There is nested sequence of subcategories: $\mathbf{U}_{cyc} \subset \mathbf{U}_0 \subset \mathbf{U}$:

- $\mathbf{U}_0$ are the simply connected graphs in $\mathbf{U}$–$\mathbf{U}_0$ corresponds to augmented cyclic operads.
- $\mathbf{U}_{cyc}$ are the simply connected graphs with non-empty boundary –$\mathbf{U}_{cyc}$ corresponds to cyclic operads.

Exercise 1.31 (Medium)**.** The full subcategories $\mathbf{U}_0$ and $\mathbf{U}_{cyc}$ are sieves of $\mathbf{U}$. In other words if $\varphi : G \to T$ is in $\mathbf{U}$ with $T \in \mathbf{U}_0$ (respectively, $\mathbf{U}_{cyc}$) then $G \in \mathbf{U}_0$ (respectively, $\mathbf{U}_{cyc}$).

1.6. **Related Work and Future Directions.** The category $\mathbf{U}_{cyc}$ is related to other categories in the literature:

- Walde has a category Ω_{cyc} which is a non-symmetric version of $\mathbf{U}_{cyc}$. That is: $\mathbf{U}_{cyc}$ is equivalent to a category $\mathbf{U}'_{cyc}$ in which every object has a specified cyclic ordering and Ω_{cyc} is the wide subcategory of $\mathbf{U}'_{cyc}$ where maps preserve the ordering.
- There is another category of Segal cyclic operads Ξ in [HRY19]. This category has the same objects as $\mathbf{U}_{cyc}$ but slightly different morphisms.
- There are Quillen adjunctions

$$\mathbf{sSet}^{\Xi^{op}} \Longleftrightarrow \mathbf{sSet}^{\mathbf{U}_{cyc}^{op}} \Longleftrightarrow \mathbf{sSet}^{\Omega^{op}}.$$

- Work of Barwick [Bar10], Hirschhorne and Volic [HV19] characterizes when $F : \mathbb{R} \to \mathbb{S}$ between **strict** Reedy categories result in Quillen adjunctions between diagram categories. It would be really nice to have such a characterisation for generalized Reedy categories so that all of these comparisons would be straightforward.

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School of Mathematics and Statistics, The University of Melbourne, Melbourne, Victoria, Australia

Email address: marcy.robertson@unimelb.edu.au