WHEELED PROPS AND CIRCUIT ALGEBRAS

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Abstract. Wheeled PROPs are operadic sturctures, originally introduced by Markl, Merkulov, and Shadrin as a generalization of categories in which morphisms which have n inputs, m outputs and are equipped with a contraction, or "trace," operation. These structures usually arise in deformation theory and the Batalin-Vilkovisky quantization formalism of theoretical physics. Circuit algebras, introduced by Bar-Natan and the first author, are a generalisation of Vaughan Jones's planar algebras, where one drops the planarity condition on "connection diagrams". They provide a useful language for the study of virtual and welded tangles in low-dimensional topology. In this note we present the two definitions – parts of which are incomplete in the literature – and we prove that wheeled PROPs and circuit algebras are equivalent mathematical objects - that is, there is an equivalence of categories. We provide a short treatment of further variants, such as undirected and coloured versions, to enable easier conversation between the low-dimensional topology and homotopy theory communities.

1. Introduction

Jones [Jon99] introduced the notion of a planar algebra as an axiomatization of the standard invariant of a finite index subfactor. A *planar algebra* is a sequence of k-vector spaces $\mathsf{V}[0], \mathsf{V}[1], \ldots, \mathsf{V}[n], \ldots$ called *box spaces* together with an action of planar tangles [HPT16, Definition 2.1]. Here, k is a field of characteristic zero.

Loosely speaking, a planar tangle is a 1-dimensional submanifold of a "disc with r holes", where the boundary points of the tangle lie on the boundary circles of the disc with holes. The boundary circles are numbered from 0 to r, with the outer circle labeled zero. For a visual example, see Figure 1.

Each such planar tangle T acts as a linear map

$$\mathsf{P}(T) : \mathsf{V}[k_1] \otimes \cdots \otimes \mathsf{V}[k_r] \longrightarrow \mathsf{V}[k_0],$$

where the number of tangle endpoints on the i-th boundary circle is k_i. Moreover, planar tangles can be *composed* by gluing the outer circle of one tangle into an inner circle of another, as long as the tangle endpoints match. The maps $P(T)$ are compatible with this composition.

The notion of a *circuit algebra* was introduced by Bar-Natan and the first author in [DBN17, Section 2], to provide a convenient formal language for virtual and welded tangles in low-dimensional topology.

A circuit algebra is a generalization of a planar algebra in which one replaces the action by planar tangles with an action by *wiring diagrams.* Wiring diagrams are similar to planar tangles but embedded submanifolds are replaced with abstract 1-manifolds whose boundary points are identified with points on the boundary circles.

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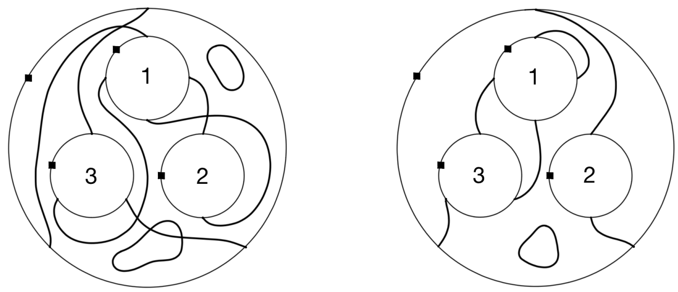


FIGURE 1. A wiring diagram on the left, a planar tangle on the right.

As a result, one obtains a purely combinatorial notion, as opposed to a topological one. We define wiring diagrams in detail and discuss this distinction in Section 2.

Just like a planar algebra, a circuit algebra is then a sequence of k-vector spaces $\mathsf{V}[0], \mathsf{V}[1], \dots, \mathsf{V}[n], \dots$ together with an action by wiring diagrams D:

$$\mathsf{P}(D) : \mathsf{V}[k_1] \otimes \cdots \otimes \mathsf{V}[k_r] \longrightarrow \mathsf{V}[k_0],$$

Like planar tangles, wiring diagrams can be composed (see Figure 3 for a quick preview), and their action is compatible with this composition.

Note that in the body of the paper we will focus on *oriented* circuit algebras, however this introduction presented the unoriented case, to keep language and notation simple.

The purpose of this paper is to give an alternative, equivalent, description of circuit algebras in terms of objects that more naturally arise in algebra, physics and homotopy theory, namely *wheeled PROPs*. A PROP can be colloquially described as a generalization of a category where morphisms have n inputs and m outputs. An illustrative example is the Segal PROP whose morphisms are elements of the moduli space of complex Riemann surfaces bounding $m+n$ labeled nonoverlapping holomorphic punctures. Composition is given by sewing Riemann surfaces together along boundaries [**?**]. This algebraic structure allowed Segal to describe certain varieties of field theories as "representations"[1] of the PROPs whose morphisms are cobordisms.

Wheeled PROPs were introduced by Markl, Merkulov, and Shadrin [MMS09] as a generalization of categories in which morphisms which have n inputs, m outputs and are equipped with a contraction, or "trace," operation. These structures usually arise in deformation theory and the Batalin-Vilkovisky quantization formalism of theoretical physics.

The main theorem of this paper establishes that the category of circuit algebras is equivalent to the category of k-linear wheeled props. As an important example we describe a linear wheeled prop of virtual tangles and a wheeled prop of welded tangles. These examples are the key to a larger project exploring the actions of homotopy automorphisms of (completions of) these wheeled props on the set of solutions to the Kashiwara-Vergne equations of Lie theory.

[1]In the literature a one uses the term algebra over a PROP for what is intuitively a representation of the PROP.

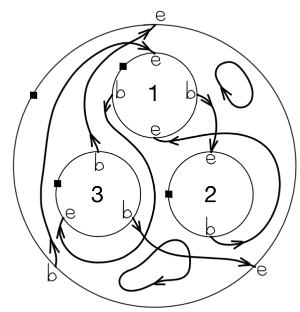


FIGURE 2. An example of an oriented (topological) wiring diagram: the manifold is drawn in thick lines, the beginning marked points are marked with “b”,the ending marked points are marked with “e”, and squares denote the base points.

The paper is organized as follows. Section 2 gives the definition of circuit algebra. Section 3 gives the definition of a wheeled PROP. Section 4 proves that the two notions are equivalent.

Remark 1.1. Both planar algebras and circuit algebras can described as *algebras over an operad*. The collection of all planar tangles forms a *topological coloured operad*, and planar algebras can be defined as algebras over this operad: this was already known in [Jon99]. Giving a similar description of circuit algebras is a simple exercise along the same lines, and it is not the point we are making. On the other hand, the fact that circuit algebras, in and of themselves, form an operadic structure – a wheeled PROP – is a crucial difference between planar and circuit algebras. Explaining this fact is the goal of this paper.

2. Circuit algebras

A circuit algebra is an algebraic structure similar in flavour to Jones's planar algebras [Jon99]. The operations of a circuit algebra are parametrized by *wiring diagrams* similar to how planar algebra operations are parametrized by planar tangles.

A key difference between planar tangles and wiring diagrams is that while planar tangles are inherently topological objects, wiring diagrams are purely combinatorial. Nonetheless, to emphasise the similarity with planar algebras and provide intuition, we present a topologically flavoured definition first, along the lines of the definition of planar tangles in [HPT16, Definition 2.1].

Both planar algebras and circuit algebras come in oriented and un-oriented versions. In what follows we focus on *oriented circuit algebras*, as they are most useful in examples and applications. Therefore, operations are parametrized by *oriented wiring diagrams*.

A *(topological) oriented wiring diagram* $D = (S, \mathcal{A}, M, f)$, as shown in Figure 2, consists of the following data:

(1) A “disc with holes”:

$$S = D_0 \setminus (\mathring{D}_1 \cup \mathring{D}_2 \cup \ldots \cup \mathring{D}_r), \quad r \geq 0.$$

Here $D_0 = \{z \in \mathbb{R}^2 \mid |z| \leq 1\}$ is a fixed *outer disk* and the $D_i = \{z \in D_0 \mid |z - a_i| \leq r_i\}$ are closed, disjoint, ordered disks in the interior of D_0. Let $\partial S = \bigcup_{i=0}^{r} \partial D_i$ denote the collection of the boundary circles of the disks comprising S. The boundary components $\partial D_1, \ldots, \partial D_r$ are called *input circles*. The boundary of the outer disk ∂D_0 is called the *output circle*. Wiring diagrams are considered unchanged by variations in the centres and sizes of input circles, as long as they remain disjoint.

(2) A finite set A_i of equidistant points is *marked* on each circle ∂D_i for $i = 0, 1, ..., r$. The marked points are partitioned into *beginning* and *ending* sets: $A_i = A_i^{\mathrm{b}} \sqcup A_i^{\mathrm{e}}$. Denote $\mathcal{A} = (A_0; A_1, ..., A_r)$, this is sometimes called the *type* of a wiring diagram. Along each circle there is a *base point*, positioned halfway between two marked points: this endows the sets of marked points with a total ordering, reading clockwise from the base point.

Should we drop the word type?

(3) An oriented compact 1-manifold with boundary M, regarded up to orientation-preserving homeomorphism.

(4) A bijection $\partial M \xrightarrow{f} \bigcup_{i=0}^{k} A_i$, such that for each connected component of M which is homeomorphic to an oriented interval I, the boundary point at the beginning of I is identified with a *beginning* marked point, and the one at the end is identified with and *ending* marked point.

Note that the key difference from planar tangles is that the manifold M is *not embedded* in S. An example of a topological oriented wiring diagram is shown in Figure 2.

Two wiring diagrams $D = (S, \mathcal{A}, M, f)$ and $D' = (T, \mathcal{B}, N, g)$, with $\mathcal{A} = (A_0; A_1, \ldots, A_r)$ and $\mathcal{B} = (B_0; B_1, \ldots, B_s)$, can be *composed*[2] along the i-th input of D if they are *"compatible"*.

The composition $D \circ_i D'$ is obtained by shrinking the disc T, and inserting it into the i^{th} input disc D_i of the disc S, lining up the base points. Glue the manifolds M and N along the identification with the marked points on D_i and D'_0, which are lined up with matching orientations if D and D' are compatible. In other words, D and D' are compatible, if $|A_i| = |B_0|$, and the canonical identification of ordered sets identifies A_i^{b} with B_0^{e} and A_i^{e} with B_0^{b}.

The identifications f and g then combine to identify the boundary of the resulting manifold with the remaining marked points – we'll describe this in more precise terms shortly.

The composition $D \circ_i D'$ is of type $(A_0; A_1, ..., A_{i-1}, B_1, ..., B_s, A_{i+1}, ..., A_r)$. This composition is illustrated in Figure 3.

Note that in presenting the definition of an oriented wiring diagram as above, one needs to take care with the positioning of input circles, marked points and base points, and then declare all of this information to be irrelevant. For planar tangles – which, by virtue of being planar objects, are inherently topological – such logistics are a necessary evil. But for oriented wiring diagrams, one may as well dispense with the discs and circles and talk only of the sets (of marked points) A_i. In light of this, we arrive at the following simplified definition:

Definition 2.1. An *oriented wiring diagram* $D = (\mathcal{A}, M, f)$ consists of:

[2] The collection of all topological oriented wiring diagrams admits the structure of a topological coloured operad, with this operadic composition.

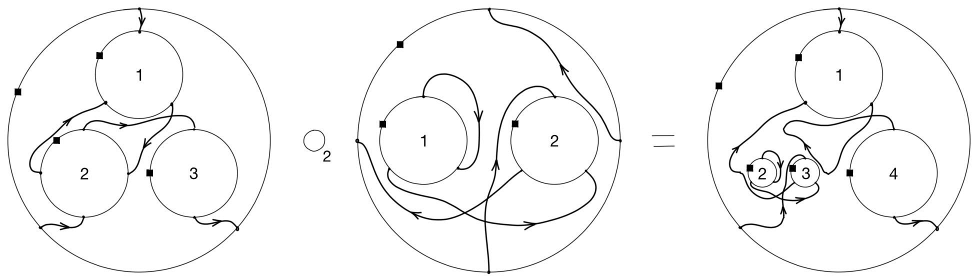


FIGURE 3. An example of oriented wiring diagram composition.

(1) A set $\mathcal{A} = \{A_0, A_1, \ldots, A_r\}_{r\in\mathbb{Z}_{\geq 0}}$ of disjoint ordered finite sets partitioned $A_i = A_i^{\mathrm{b}} \sqcup A_i^{\mathrm{e}}$, considered up to order and partition preserving isomorphisms. The elements of the sets A_i^{b} are referred to as *beginning labels* and the elements of A_i^{e} are *ending labels*. The set A_0 plays a distinguished role: its elements are called *output labels*, while the sets $A_1, ..., A_r$ contain *input labels*.
(2) An oriented compact 1-manifold with boundary M, regarded up to orientation-preserving homeomorphism. Let ∂M^{b} denote the set of "beginning" boundary points of M, and ∂M^{e} the set of "ending" boundary points. (A connected compact oriented 1-manifold with boundary is homeomorphic to an oriented interval with one beginning and one ending point.)
(3) A bijection $f : \partial M \to \cup_{i=0}^{r} A_i$, so that $\partial M^{\mathrm{b}} \xrightarrow{f} \bigcup_{i=0}^{r} A_i^{\mathrm{b}}$, $\partial M^{\mathrm{e}} \xrightarrow{f} \bigcup_{i=0}^{r} A_i^{\mathrm{e}}$.

Given two wiring diagrams

$$D = (\{A_0, A_1, \ldots, A_r\}, M, f), \quad D' = (\{B_0, B_1, \ldots, B_s\}, N, g),$$

the *composition* $D \circ_i D'$ for $0 \leq i \leq r$ is defined whenever $|A_i| = |B_0|$, and for the unique order preserving bijection $\psi : A_i \to B_0$, $\psi(A_i^{\mathrm{b}}) = B_0^{\mathrm{e}}$ and $\psi(A_i^{\mathrm{e}}) = B_0^{\mathrm{b}}$. We may assume that all of the sets $\{A_i, B_j\}_{i=0,\ldots,r; j=0,\ldots,s}$ are pairwise disjoint: if they are not, apply an order preserving relabelling to the sets $\{B_j\}_{j=0,\ldots,s}$ to make them so.

Then, the wiring diagram $D \circ_i D'$ is defined to be

$$D \circ_i D' = (\{A_0, A_1, \ldots, A_{i-1}, B_1, \ldots, B_s, A_{i+1}, \ldots, A_r, \}, M \sqcup_\varphi N, f \sqcup_\varphi g),$$

were the map φ is the identification of the boundary points in $f^{-1}(A_i)$ and $g^{-1}(B_0)$ given by:

$$\partial M \supseteq f^{-1}(A_i) \xrightarrow{f} A_i \xrightarrow{\psi} B_0 \xrightarrow{g^{-1}} g^{-1}(B_0) \subseteq \partial N.$$

The manifold $M \sqcup_\varphi N$ is the 1-manifold obtained by gluing M and N along φ. The map $f \sqcup_\varphi g$ is f on $\partial M \setminus f^{-1}(A_i)$, and g on $\partial N \setminus g^{-1}(B_0)$.

Remark 2.2. The oriented 1-manifold M in Definition 2.1 is compact with boundary, hence it is a disjoint union of a finite number of oriented intervals and circles. The endpoints of the intervals are identified with the labels, and as such, the only role of the intervals is to define a perfect matching on the set of labels. So we could define a wiring diagram as a triple (A, p, l), where A is as above, p is a perfect matching between $\bigsqcup_{i=0}^{r} A_i^{\mathrm{b}}$ and $\bigsqcup_{i=0}^{r} A_i^{\mathrm{e}}$, and $l \in \mathbb{Z}_{\geq 0}$ is a non-negative integer, "the number of circles in M".

This definition through perfect matchings makes the purely combinatorial nature of oriented wiring diagrams most apparent, and indeed we will use it as an alternative notation for wiring diagrams. The drawback of doing away with the 1-manifolds is in the definition the composition of wiring diagrams, in particular the perfect matching and number of circles resulting from a composition. This is possible, but a headache (we encourage the reader to try), and hence we stick with the much more elegant definition of composition through gluing 1-manifolds.

From here on, we denote the cardinality of the sets A_i with lower case bold letters, i.e. $|A_i| = \mathbf{a}_i$, $|A_i^{\mathrm{b/e}}| = \mathbf{a}_i^{\mathrm{b/e}}$, $|B_j^{\mathrm{b/e}}| = \mathbf{b}_j^{\mathrm{b/e}}$. The set A_i is an ordered set partitioned into sets of *beginning* and *ending* labels. Let $\underline{\epsilon}_i$ denote the sign vector of length $\mathbf{a}_i$ consisting of a negative sign for each beginning and a positive sign for each outgoing label. For example, in the wiring diagram of Figure 2, $\underline{\epsilon}_0 = (+,+,-)$, $\underline{\epsilon}_1 = (+,-,+,-)$, $\underline{\epsilon}_2 = (+,-)$, and $\underline{\epsilon}_3 = (-,-,+)$.

The following definition of an oriented circuit algebra is based on [DBN17, Definition 2.10]:

Definition 2.3. An **oriented circuit algebra** consists of a collection of vector spaces $\{\mathsf{V}[n,m]\}_{n,m\in\mathbb{Z}_{\geq 0}}$ and linear maps between them parametrized by oriented wiring diagrams. Namely, for each wiring diagram $D = (A, M, f)$, there is a corresponding linear map $F_D : \mathsf{V}[\mathbf{a}_1^{\mathrm{b}}, \mathbf{a}_1^{\mathrm{e}}] \otimes \ldots \otimes \mathsf{V}[\mathbf{a}_r^{\mathrm{b}}, \mathbf{a}_r^{\mathrm{e}}] \to \mathsf{V}[\mathbf{a}_0^{\mathrm{b}}, \mathbf{a}_0^{\mathrm{e}}]$. This data must satisfy the following axioms:

(1) The composition of wiring diagrams corresponds to composition of linear maps in the following sense. Let
$$D = (\{A_0, \ldots, A_r\}, M, f), \quad D' = (\{B_0, \ldots, B_s\}, N, g)$$
be two wiring diagrams compatible for composing as $D \circ_i D'$. Then the map of vector spaces corresponding to the composition $D \circ_i D'$ is
$$F_{D\circ_i D'} = F_D \circ (\mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes F_{D'} \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}).$$

(2) There is an action of the symmetric groups $\mathcal{S}_r$ on wiring diagrams with r input sets (or input discs), which permutes the order of the input sets. The assigment of linear maps to wiring diagrams is equivariant under this action in the following sense: let $D = (\{A_0, A_1 \ldots, A_r\}, M, f)$ be a wiring diagram, $\sigma \in \mathcal{S}_r$, and
$$\sigma D = (\{A_0, A_{\sigma(1)} \ldots, A_{\sigma(k)}\}, M, f)$$
be the same wiring diagram with the input sets re-ordered.[3] Then the induced linear map $F_{\sigma D}$ is $F_D \circ \sigma^{-1}$, where σ^{-1} acts on
$$\mathsf{V}[\mathbf{a}_{\sigma(1)}^{\mathrm{b}}, \mathbf{a}_{\sigma(1)}^{\mathrm{e}}] \otimes \ldots \otimes \mathsf{V}[\mathbf{a}_{\sigma(r)}^{\mathrm{b}}, \mathbf{a}_{\sigma(r)}^{\mathrm{e}}]$$
by permuting the tensor factors.

A *morphism* of circuit algebras $\Phi : \mathsf{V} \to \mathsf{W}$ is a family of linear maps
$$\{\Phi_{n,m} : \mathsf{V}[n,m] \to \mathsf{W}[n,m]\}_{n,m\geq 0}$$
which commutes with the circuit algebra structure. The category of all circuit algebras is denoted CA.

Remark 2.4. For every sequence $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_t)$, $t \geq 0$, where $\epsilon_i = \pm 1$, there is an *identity wiring diagram* $\mathrm{Id}_{\underline{\epsilon}} = (A, I, f)$, where:

[3] Note that the output set A_0 is fixed.

- $A = \{A_0; A_1\}$ with $A_0 = \{1, 2, ...t\}$, $A_1 = \{t+1, ..., 2t\}$; and
  - ➔ $A_0^{\mathrm{b}} = \{1 \leq i \leq t : \epsilon_i = +1\}$,
  - ➔ $A_0^{\mathrm{e}} = \{1 \leq i \leq t : \epsilon_i = -1\}$,
  - ➔ $A_1^{\mathrm{b}} = \{t + i : 1 \leq i \leq t, \epsilon_i = -1\}$,
  - ➔ $A_1^{\mathrm{e}} = \{t + i : 1 \leq i \leq t, \epsilon_i = +1\}$.
- $I = \sqcup_{j=1}^{t} I_j$, where each I_j is a unit interval with increasing orientation if $\epsilon_j = +1$ and decreasing orientation if $\epsilon_j = -1$;
- f is defined on $\partial I_j = \{0^j, 1^j\}$ by $f(0^j) = j \in A_0, f(1^j) = t + j \in A_1$.

The corresponding linear map $F_{\mathrm{Id}_{\underline{\epsilon}}} : \mathsf{V}[n,m] \to \mathsf{V}[n,m]$ is the identity on elements of $V[n,m]$ with matching orientation $\underline{\epsilon}$, and is zero otherwise. (Alternatively $\mathrm{Id}_{\underline{\epsilon}} = (A, p, 0)$, where the matching p is the canonical order-preserving bijection $A_0 \to A_1$.)

Need to define the orientations on V[n,m]

Remark 2.5. While we have defined a circuit algebra in terms of vector spaces and linear maps, the definition holds in any closed, symmetric monoidal category.

Example 2.6. Oriented circuit algebras were introduced to provide a convenient language for the study of virtual and welded tangles in knot theory. Knots have a combinatorial description in terms of *knot diagrams*, and two knot diagrams represent isotopic knots if and only if they are equivalent through a sequence of relations called *Reidemeister moves*. Virtual knots were introduced by Kauffman in [**?**] as a combinatorial generalisation of knots, and were later shown to represent knots in thickened surfaces modulo stabilisation (that is, the addition or deletion of empty handles).

Virtual tangles are a similar generalisation of classical tangles, usually given combinatorially in terms of generators and relations. Oriented virtual tangles $v\mathcal{T}$ form a circuit algebra, where $V[n,n] = \{$the space of tangles with n open strands$\}$, $V[n,m] = \{0\}$ if $n \neq m$, and the wiring diagrams act by concatenation of strands. In this case.

Add references

Needs to be expanded.

Ordinary tangles form a planar algebra, and originally virtual tangles were described as a planar algebra, which is obtained from tangles by adding a new type of crossing (the *virtual crossing*) and many new relations which describe how virtual crossings interact with classical crossings and each other. One advantage of the notion of a circuit algebra is that it describes virtual crossings using exactly the same generators and relations as ordinary tangles, changing only the algebraic structure in which the generating occurs.

INSERT PICTURE: example of a virtual tangle

INSERT PICTURE: Reidemeister moves for virtual tangles

3. Wheeled Props

Operadic structures provide the algebraic tools and context for studying “spaces of operations”, regardless of what those operations act on. They come in many flavours depending on the number of inputs and outputs that operations may have, and the types of compositions allowed. PROPs are among the oldest operadic structures.

The notion of a PROP was introduced by MacLane in 1963 in order to encode axioms for operations with multiple inputs and outputs. Wheeled PROPs in particular arise naturally in geometry, deformation theory and theoretical physics. For example, in the Batalin-Vilkovisky quantization formalism, solution spaces of certain classes of important highly non-linear differential equations are controlled by wheeled PROPs, viewed as resolutions of very compact graphical data.

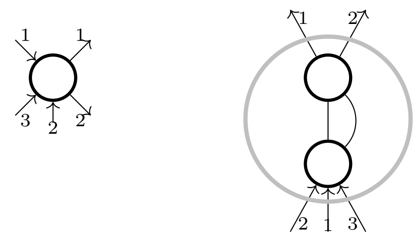


FIGURE 4. Elements of $\mathsf{V}[2;3]$. If we do not label the legs, we will often assume a canonical ordering, starting from the upper left corner and proceeding clockwise, as in the example on the right we have $1^{in}, 1^{out}, 2^{out}, 2^{in}, 3^{in}$.

The goal of this section is to give a detailed definition of wheeled PROPs, with a view towards matching wheeled PROP *compositions* to circuit algebra *wiring diagrams*.

Definition 3.1. An $\mathcal{S}$-*bimodule* is a family of vector spaces $\{\mathsf{V}[m,n]\}_{m,n\geq 0}$, and for each $\mathsf{V}[m,n]$, commuting actions of the symmetric groups $\mathcal{S}_m$ on the left and $\mathcal{S}_n$ on the right. (In other words, an action of $\mathcal{S}_m^{op} \times \mathcal{S}_n$.)

Remark 3.2. An alternative description of an $\mathcal{S}$-bimodule is to declare that $\mathsf{V}[m;n]$ is spanned by decorated, directed graphs with half-edges, or legs (See, for example, [?] around Definition 2.1.2). Some of the legs are called inputs and are labelled with an ordered set of n labels, up to order-preserving re-labeling; and $\mathcal{S}_n$ acts by permuting the legs, or equivalently, the labels. The rest of the legs are outputs, and are labelled with an ordered set of m-labels. One could draw the graphs in a disc, with nodes denoted by small circles, and the legs arranged, in order, around the boundary of the disc, as shown in Figure 4. (The small circles in these pictures do not correspond with the input discs of circuit algebras but vertices in a graph.)

Definition 3.3. A **wheeled prop** $\mathsf{V} := \left(\mathsf{V}, {}_i\circ_j, \circ_h, \xi_i^j\right)$ consists of:

- An $\mathcal{S}^{\uparrow}$-bimodule $\mathsf{V} = \{\mathsf{V}[\underline{m};\underline{n}]\}$
- A *horizontal composition*
$$\circ_h : \mathsf{V}[\underline{d};\underline{c}] \otimes \mathsf{V}[\underline{b};\underline{a}] \longrightarrow \mathsf{V}[\underline{d},\underline{b};\underline{c},\underline{a}]$$
- A morphism
$$I_{\emptyset} : \mathbf{k} \longrightarrow \mathsf{V}[0;0]$$
called the *empty unit*.
- A *contraction* operation
$$\mathsf{V}[\underline{d};\underline{c}] \xrightarrow{\xi_i^j} \mathsf{V}[\underline{d} \setminus \{d_j\}; \underline{c} \setminus \{c_i\}]$$
for some $j \leq |\underline{d}|$, $i \leq |\underline{c}|$.
- A morphism
$$I : \mathbf{k} \longrightarrow \mathsf{V}[1;1]$$
called the *unit*.

The horizontal composition and contraction operations combine to give an additional *dioperadic composition*

$$\begin{tikzcd}
\mathsf{V}[\underline{d};\underline{c}] \otimes \mathsf{V}[\underline{c};\underline{b}] \arrow[rr, "{}_i\circ_j"] \arrow[dr] & & \mathsf{V}[\underline{d};\underline{b}] \\
& \mathsf{V}[\underline{d},\underline{c};\underline{c},\underline{b}] \arrow[ur] &
\end{tikzcd}$$

for some $0 \leq i \leq |\underline{b}|$, $0 \leq j \leq |\underline{d}|$. The horizontal composition, dioperadic composition and contraction are all associative and equivariant. The horizontal and dioperadic compositions are unital.

To elaborate, the horizontal composition is associative, unital and equivariant in the following sense that they satisfy the following axioms.

H1 The horizontal composition is associative in the sense that

$$\begin{tikzcd}
\mathsf{V}[f;e] \otimes \mathsf{V}[d;c] \otimes \mathsf{V}[b;a] \arrow[r, "\circ_h \otimes id"] \arrow[d, "id\otimes\circ_h"'] & \mathsf{V}[f,d;e,c] \otimes \mathsf{V}[b;a] \arrow[d, "\circ_h"] \\
\mathsf{V}[f;e] \otimes \mathsf{V}[d,b;c,a] \arrow[r, "\circ_h"] & \mathsf{V}[f,d,b;e,c,a]
\end{tikzcd}$$

commutes.

H2 The horizontal composition is bi-equivariant in the sense that the following diagrams

$$\begin{tikzcd}
\mathsf{V}[\underline{d};\underline{c}] \otimes \mathsf{V}[\underline{b};\underline{a}] \arrow[r, "\circ_h"] \arrow[d, "(\tau_1;\sigma_1)\otimes(\tau_2;\sigma_2)"'] & \mathsf{V}[\underline{d},\underline{b};\underline{c},\underline{a}] \arrow[d, "(\tau_1\times\tau_2;\sigma_1\times\sigma_2)"] \\
\mathsf{V}[\sigma_1\underline{d};\underline{c}\tau_1] \otimes \mathsf{V}[\sigma_2\underline{b};\underline{a}\tau_2] \arrow[r, "\circ_h"] & \mathsf{V}[\sigma_1\underline{d},\sigma_2\underline{b};\underline{c}\tau_1,\underline{a}\tau_2]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{V}[\underline{d};\underline{c}] \otimes \mathsf{V}[\underline{b};\underline{a}] \arrow[r, "\circ_h"] \arrow[d, "swap"'] & \mathsf{V}[\underline{d},\underline{b};\underline{c},\underline{a}] \arrow[d, "(\tau;\sigma)"] \\
\mathsf{V}[\underline{b};\underline{a}] \otimes \mathsf{V}[\underline{d};\underline{c}] \arrow[r, "\circ_h"] & \mathsf{V}[\underline{b},\underline{d};\underline{a},\underline{c}]
\end{tikzcd}$$

are commutative for all permutations $\sigma_1 \in \mathcal{S}_d$, $\sigma_2 \in \mathcal{S}_b$, $\tau_1 \in \mathcal{S}_c$, $\tau_2 \in \mathcal{S}_a$ and all block permutations $\sigma = (12) < d, b > \in \mathcal{S}_{d+b}$ and $\tau = (12) < a, c > \in \mathcal{S}_{c+a}$.

H3 The empty unit is a two-sided unit for the horizontal composition in the sense that the following diagram commutes

$$\begin{tikzcd}
& \mathsf{V}[\underline{d};\underline{c}] \arrow[dl] \arrow[dr] \arrow[ddd] & \\
\mathbf{k} \otimes \mathsf{V}[\underline{d};\underline{c}] \arrow[d] & & \mathsf{V}[\underline{d};\underline{c}] \otimes \mathbf{k} \arrow[d] \\
\mathsf{V}[0;0] \otimes \mathsf{V}[\underline{d};\underline{c}] \arrow[dr] & & \mathsf{V}[\underline{d};\underline{c}] \otimes \mathsf{V}[0;0] \arrow[dl] \\
& \mathsf{V}[\underline{d};\underline{c}] &
\end{tikzcd}$$

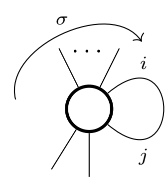
The contraction operation is commutative and bi-equivariant in the following sense:

C1 Contraction is bi-equivariant in the following sense.

$$\begin{tikzcd} \mathsf{V}[\underline{d};\underline{c}] \arrow[r, "\xi_i^j"] \arrow[d] & \mathsf{V}[\underline{d}\setminus\{d_i\};\underline{c}\setminus\{c_j\}] \arrow[d] \\ \mathsf{V}[\sigma\underline{d};\underline{c}\tau] \arrow[r, "\xi_j^i"] & \mathsf{V}[\sigma\underline{d}\setminus\{d_i\};\underline{c}\tau\setminus\{c_j\}] \end{tikzcd}$$

where $\sigma \in \mathcal{S}_d$, $\tau \in \mathcal{S}_c$ and

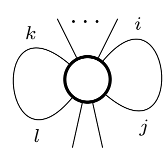
$$(\sigma\underline{d}\setminus\{d_i\};\underline{c}\tau\setminus\{c_j\}) = (d_{\sigma(1)},\dots,\hat{d}_j,\dots,d_{\sigma(m)};c_{\tau^{-1}(1)},\dots,\hat{c}_j,\dots,c_{\tau^{-1}(n)}).$$



C2 Contraction is commutative in the following way. If $|\underline{d}| = m \geq 2$, $|\underline{c}| = n \geq 2$, then ξ_j^i and ξ_l^k commute in the following way

$$\begin{tikzcd} \mathsf{V}[\underline{d};\underline{c}] \arrow[r, "\xi_i^j"] \arrow[d, "\xi_k^l"'] & \mathsf{V}[\underline{d}\setminus\{d_i\};\underline{c}\setminus\{c_j\}] \arrow[d, "\xi_k^l"] \\ \mathsf{V}[\underline{d}\setminus\{d_k\};\underline{c}\setminus\{c_l\}] \arrow[r, "\xi_i^j"'] & \mathsf{V}[\underline{d}\setminus\{d_i,d_k\};\underline{c}\setminus\{c_j,c_l\}] \end{tikzcd}$$

commutes.



Contraction and horizontal composition commute in the following way.

HC1 The square

$$\begin{tikzcd} \mathsf{V}[\underline{d};\underline{c}]\otimes\mathsf{V}[\underline{b};\underline{a}] \arrow[r, "\circ_h"] \arrow[d, "\xi_j^i\otimes id"'] & \mathsf{V}[\underline{d},\underline{b};\underline{c},\underline{a}] \arrow[d, "\xi_j^i"] \\ \mathsf{V}[\underline{d}\setminus\{d_i\};\underline{c}\setminus\{c_j\}]\otimes\mathsf{V}[\underline{b};\underline{a}] \arrow[r, "\circ_h"] & \mathsf{V}[\underline{d}\setminus\{d_i\},\underline{b};\underline{c}\setminus\{c_j\},\underline{a}] \end{tikzcd}$$

commutes whenever $i \leq |\underline{d}|$, $j \leq |\underline{c}|$.

HC2 The square

$$\begin{array}{ccc} \mathsf{V}[\underline{d};\underline{c}] \otimes \mathsf{V}[\underline{b};\underline{a}] & \xrightarrow{\quad\circ_h\quad} & \mathsf{V}[\underline{d},\underline{b};\underline{c},\underline{a}] \\ {\scriptstyle \xi^i_j \otimes id}\downarrow & & \downarrow{\scriptstyle \xi^i_j} \\ \mathsf{V}[\underline{d};\underline{c}] \otimes \mathsf{V}[\underline{b}\setminus\{b_k\};\underline{a}\setminus\{a_l\}] & \xrightarrow{\circ_h} & \mathsf{V}[\underline{d},\underline{b}\setminus\{b_k\};\underline{c},\underline{a}\setminus\{a_l\}] \end{array}$$

is commutative whenever $k \leq |\underline{b}|$, $l \leq |\underline{a}|$.

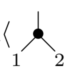
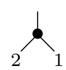
Similar axioms can be derived for the *dioperadic composition*. For example, the dioperadic composition is commutative in equivariant. As just one example, given $\sigma \in \mathcal{S}_{|\underline{d}|}$, $\tau \in \mathcal{S}_{|\underline{c}|}$ and $\pi \in \mathcal{S}_{|\underline{b}|}$ the dioperadic composition is biequivariant in the sens that

$$\begin{array}{ccc} \mathsf{V}[\underline{d};\underline{c}] \otimes \mathsf{V}[\underline{c};\underline{b}] & \xrightarrow[{}_i\circ_j]{} & \mathsf{V}[\underline{d};\underline{b}] \\ {\scriptstyle (\tau^{-1};\sigma)\otimes(\pi;\tau)}\downarrow & & \downarrow{\scriptstyle (\pi;\sigma)} \\ \mathsf{V}[\sigma\underline{d};\underline{c}\tau^{-1}] \otimes \mathsf{V}[\underline{c};\underline{b}] & \xrightarrow[p]{} & \mathsf{V}[\sigma\underline{d};\underline{b}\pi] \end{array}$$

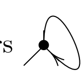
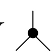
commutes. For a complete accounting of these additional axioms see [YJ15, Chapter 10].

Definition 3.4. A *morphism* $\mathsf{V} \xrightarrow{f} \mathsf{W}$ of wheeled props is a map of the underlying $\mathcal{S}^{\uparrow}$-bimodules that respect the symmetric group actions, horizontal composition and contractions. The category of all wheeled props is denoted by wProp.

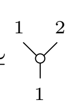
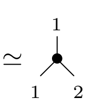
Example 3.5. There is a wheeled operad of associative algebras Ass defined as

$\mathsf{Ass}[m;n] =$ span$\langle$  ,  $\rangle$ if $[m;n] = [1;2]$; 0 otherwise

which parameterizes associative algebras ([]).

The contraction ξ on one of the generators  $\in \mathsf{Ass}[0;1]$ represents a multiplication of a finite dimensional vector space V  in $Hom(V \otimes V, V)$ under a natural trace map $Hom(V \otimes V, V) \to Hom(V, \mathbf{k})$.

Example 3.6. A *Lie n-bialgebra* on a graded vector space V is a pair of linear maps,

$\Delta \simeq$  $: \mathsf{V} \to \mathsf{V} \wedge \mathsf{V}, \quad [\bullet] \simeq$  $: \wedge^2(\mathsf{V}[-n]) \to V[-n]$

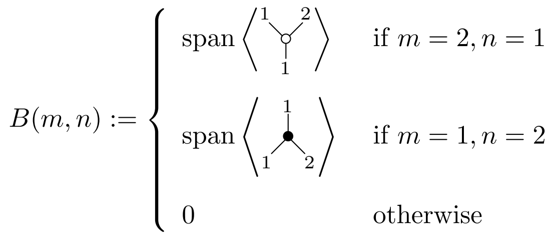
making the space V into a Lie coalgebra and the space $\mathsf{V}[-n]$ into a Lie algebra and satisfying, for any $a, b \in \mathsf{V}$, the compatibility condition

$$\Delta[a\bullet b] = \sum a_1 \otimes [a_2 \bullet b] + [a \bullet b_1] \otimes b_2 + (-1)^{|a||b|+n|a|+n|b|}([b \bullet a_1] \otimes a_2 + b_1 \otimes [b_2 \bullet a]),$$

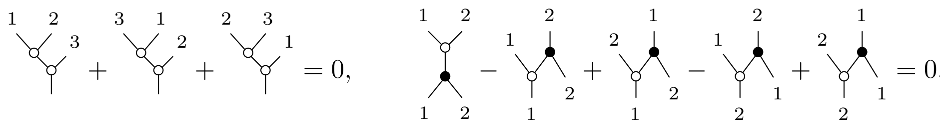
Here $\Delta a =: \sum a_1 \otimes a_2$ and $\Delta b =: \sum b_1 \otimes b_2$. The case $n = 0$ gives the notion of Lie bialgebra which was introduced by Drinfeld [**?**] in the context of quantum groups.

The case $n = 1$ is used by Merkulov in the study of Poisson geometry. The prop of Lie 1-bialgebras, LB, is the quotient of the free prop, $F^{\uparrow}\langle B \rangle$, generated by an

$\mathcal{S}^{\uparrow}$-bimodule, B



modulo the ideal generated by Jacobi relations as well as the following



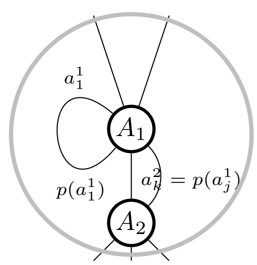
4. Equivalence

4.1. **Wheeled Props from Wiring Diagrams.** The point of this section is to construct an equivalence of categories from the category of circuit algebras to the category of wheeled props. To start, we will note that given an $\mathcal{S}^{\uparrow}$-bimodule $\mathsf{V} = \{\mathsf{V}[m;n]\}_{m,n\geq 0}$ we can construct wiring diagrams which give a wheeled prop operations on V. To assist in this, let us recall some notation, in a wiring diagram $D = (A, M, p)$ the set $A = \{A_0, \ldots, A_r\}$ is the set of disjoint, ordered finite sets of labels. We set $|A_i| = \mathbf{a}_i$ as in previous sections. A *k-segment* of A_i is a sub-list of the elements of A_i of length k.

Definition 4.1. A *decoration* of a wiring diagram $D = (\{D_0, D_1, \ldots, D_j\}), M, p)$ is an element of the vector space $w \in \mathsf{V}[\mathbf{d}_0^{out}; \mathbf{d}_0^{in}]$ comprised of tensors

$$w_1 \in \mathsf{V}[\mathbf{d}_1^{out}; \mathbf{d}_1^{in}], \ldots w_j \in \mathsf{V}[\mathbf{d}_j^{out}; \mathbf{d}_j^{in}].$$

Example 4.2. The bijection $p : \partial M \to \bigcup_{i=0}^{r} A_i$ gives an identification of various k-segments of the sets A_i. We will prove below that these pairings give compositions of horizontal, dioperadic and contraction operations. In the picture below, we depict some internal wires of a wiring diagram defined by the pairing p.



To be more explicit, any pairing identifies a k-segment $a_{l_1}^r, \ldots, a_{l_k}^r$ of A_r with some k-segment $a_{l_1}^s, \ldots, a_{l_k}^s$ of A_s. In the case where $A_r \neq A_s$ the pairing p induces the linear map

$$D : \mathsf{V}(\mathbf{a}_r) \otimes \mathsf{V}(\mathbf{a}_s) \longrightarrow \mathsf{V}(\mathbf{a}_r + \mathbf{a}_s - k).$$

which we can show is a composite of dioperadic compositions. In the case that $A_r = A_s$ then the pairing induces a the linear map

$$D : \mathsf{V}(\mathbf{a}_r) \longrightarrow \mathsf{V}(\mathbf{a}_r - k)$$

which we will show is a composition of contractions.

Lemma 4.3. *Let $H = (\{H_0, H_1, H_2\}, M, p)$ be a wiring diagram in which $p : \partial M \to \bigcup_{i=0}^{2} H_i$ has the property that $\partial M \cap H_1 \cap H_2 = \emptyset$. Then H parametrizes a horizontal composition of an $\mathcal{S}^{\uparrow}$-bimodule $\mathsf{V} = \{\mathsf{V}[\underline{m}; \underline{n}]\}_{m,n \geq 0}$.*

Proof. Consider the boundary points H_i, $i = 1, 2$. Our assumption that H_i has $\partial M \cap H_1 \cap H_2 = \emptyset$ tells us that the assignment

$$p : \partial(M) \longrightarrow H_0 \cup H_1 \cup H_2$$

has the property that for any component e of the 1-manifold M, if $a \in \partial(e)$ is in H_1, then the other endpoint $a' \in \partial(e)$ is either in H_1 or H_0. Similarly, for a component e' of M with $b \in \partial(e') \cap H_2$.

Write $\mathbf{h}_i^{in}$ denote the points in H_i, $i = 0, 1, 2$, which have been designated inputs. Similarly, let $\mathbf{h}_i^{out}$ denote the points in H_i, $i = 0, 1, 2$, which have been designated outputs. The wiring diagram H parameterizes a map of vector spaces

$$\mathsf{V}[\mathbf{h}_1^{in}; \mathbf{h}_1^{out}] \otimes \mathsf{V}[\mathbf{h}_2^{in}; \mathbf{h}_2^{out}] \longrightarrow \mathsf{V}[\mathbf{h}_1^{in}, \mathbf{h}_2^{in}; \mathbf{h}_1^{out}, \mathbf{h}_2^{out}] \xrightarrow[p]{=} \mathsf{V}[\mathbf{h}_0^{in}; \mathbf{h}_0^{out}]$$

which is a horizontal composition on the $\mathcal{S}$-module V. □

Lemma 4.4. *Horizontal composition as defined in Lemma 4.3 is associative, equivariant and unital.*

Proof. Let V be an $\mathcal{S}^{\uparrow}$-bimodule and let $H = (\{H_0, H_1, H_2\}, M, p)$ be a wiring diagram as in Lemma 4.3. An action on $\mathsf{V}[\mathbf{h}_i^{out}; \mathbf{h}_i^{in}]$, $i = 1, 2$ consists of a bijection of the labels in H_i, $i = 1, 2$ and, composing with the bijection p implies that the following diagram

$$\begin{tikzcd}
\mathsf{V}[\mathbf{h}_1^{out}; \mathbf{h}_1^{in}] \otimes \mathsf{V}[\mathbf{h}_2^{out}; \mathbf{h}_2^{in}] \arrow[r, "\circ_h"] \arrow[d, "(\tau_1;\sigma_1)\otimes(\tau_2;\sigma_2)"'] & \mathsf{V}[\mathbf{h}_1^{out}, \mathbf{h}_2^{out}; \mathbf{h}_1^{in}, \mathbf{h}_2^{in}] \arrow[d, "(\tau_1\times\tau_2;\sigma_1\times\sigma_2)"] \\
\mathsf{V}[\sigma_1\mathbf{h}_1^{out}; \mathbf{h}_1^{out}\tau_1] \otimes \mathsf{V}[\sigma_2\mathbf{h}_2^{out}; \mathbf{h}_2^{in}\tau_2] \arrow[r, "\circ_h"] & \mathsf{V}[\sigma_1\mathbf{h}_1^{out}, \sigma_2\mathbf{h}_2^{out}; \mathbf{h}_1^{in}\tau_1, \mathbf{h}_2^{in}\tau_2]
\end{tikzcd}$$

for all permutations $\sigma_1 \in \mathcal{S}_{\mathbf{h}_1^{out}}$, $\sigma_2 \in \mathcal{S}_{\mathbf{h}_2^{out}}$, $\tau_1 \in \mathcal{S}_{\mathbf{h}_1^{out}}$, $\tau_2 \in \mathcal{S}_{\mathbf{h}_2^{out}}$.

Similarly, the bijection p implies that swapping the order of H_1 and H_2 makes no difference in the wiring diagram, except to add some self-crossings of the manifold M. This implies that the following diagram commutes.

$$\begin{tikzcd}
\mathsf{V}[\mathbf{h}_1^{out}; \mathbf{h}_1^{in}] \otimes \mathsf{V}[\mathbf{h}_2^{out}; \mathbf{h}_2^{in}] \arrow[r, "\circ_h"] \arrow[d, "swap"'] & \mathsf{V}[\mathbf{h}_1^{out}, \mathbf{h}_2^{out}; \mathbf{h}_1^{in}, \mathbf{h}_2^{in}] \arrow[d, "(\tau;\sigma)"] \\
\mathsf{V}[\mathbf{h}_2^{out}; \mathbf{h}_2^{in}] \otimes \mathsf{V}[\mathbf{h}_1^{out}; \mathbf{h}_1^{in}] \arrow[r, "\circ_h"] & \mathsf{V}[\mathbf{h}_2^{out}, \mathbf{h}_1^{out}; \mathbf{h}_2^{in}, \mathbf{h}_1^{in}]
\end{tikzcd}$$

are commutative for all block permutations $\sigma = (12) < \mathbf{h}_1^{out}, \mathbf{h}_2^{out} > \in \mathcal{S}_{\mathbf{h}_1^{out} + \mathbf{h}_2^{out}}$ and $\tau = (12) < \mathbf{h}_1^{in}, \mathbf{h}_2^{in} > \in \mathcal{S}_{\mathbf{h}_1^{in} + \mathbf{h}_2^{in}}$.

We would like to show that the following diagram commutes.

$$\begin{tikzcd}
\mathsf{V}[\mathbf{h}_1^{out};\mathbf{h}_1^{in}]\otimes\mathsf{V}[\mathbf{j}_1^{out};\mathbf{j}_1^{in}]\otimes\mathsf{V}[\mathbf{j}_2^{out};\mathbf{j}_2^{in}] \arrow[r, "K^*\otimes id"] \arrow[d, "id\otimes J^*"'] & \mathsf{V}[\mathbf{h}_1^{out},\mathbf{j}_1^{out};\mathbf{h}_1^{in},\mathbf{j}_1^{in}]\otimes\mathsf{V}[\mathbf{j}_2^{out};\mathbf{j}_2^{in}] \arrow[d, "L^*"] \\
\mathsf{V}[\mathbf{h}_1^{out};\mathbf{h}_1^{in}]\otimes\mathsf{V}[\mathbf{j}_1^{out},\mathbf{j}_2^{in};\mathbf{j}_1^{out},\mathbf{j}_2^{in}] \arrow[r, "H^*"] & \mathsf{V}[\mathbf{h}_1^{out},\mathbf{j}_1^{out},\mathbf{j}_2^{out};\mathbf{h}_1^{in},\mathbf{j}_1^{in},\mathbf{j}_2^{in}]=\mathsf{V}[\mathbf{h}_0^{out};\mathbf{h}_0^{in}]
\end{tikzcd}$$

commutes. The equality on the bottom right is via the bijection $J_0 \to H_2$. The first composite $H^*(id\otimes J^*)$ is given by the composite of two wiring diagrams $H = (\{H_0, H_1, H_2\}, M, p)$ and $J = (\{J_0, J_1, J_2\}, N, q)$ together with a bijection $J_0 \to H_2$. The second composite is given by two wiring diagrams $K = (\{K_0, K_1, K_2\}, M', p')$ and $L = (\{L_0, L_1, L_2\}, N', q')$ with $|K_1| = |H_1|$, $|K_2| = |J_1|$, $|L_1| = |H_1| + |J_1|$ and $|L_2| = |J_2|$. To see that the diagram commutes one need only observe that both composite wiring diagrams $J\circ H$ and $L\circ K$ have the same number of points on each internal disk and the outer disk. The only real difference between the composite wiring diagrams are the number of self crossings of the respective 1-manifolds and reorderings of the boundary points.

It remains to show that the horizontal composition is unital. It suffices to construct a wiring diagram which gives an empty unit $I_\emptyset : \mathbf{k} \to \mathsf{V}[0;0]$ and show how that composes with other wiring diagrams. Set $I_\emptyset = (I_0, I_1, N, q)$ be the wiring diagram in which both the outer disk I_0 and inner disk I_1 have no labels. The one manifold N can be any closed one manifold which does not intersect the disk I_0 or I_1 and q maps the empty set to the empty set. It follows that whenever any wiring diagram can be composed with $I_\emptyset$ the same the same wiring diagram is returned. □

The next thing we show is that wiring diagrams with internal wires represent contraction operations or dioperadic compositions.

Lemma 4.5. *Let $\mathsf{V} = \{\mathsf{V}[m;n]\}_{m,n\geq 0}$ be an $\mathcal{S}^\uparrow$-bimodule and let $C = (\{C_0, C_1\}, N, q)$ be a wiring diagram which has a unique internal wire. Then C parametrizes a contraction operation as in Definition 3.3.*

Proof. Let $\mathbf{c}_i$ denote the boundary points $C_i \cap \partial N$ with $i = 0, 1$. Let e denote the component of N forming an internal wire with $\partial(e) \in C_1$. We may assume, without loss of generality, that $a \in \partial(e)$ is designated an input and $a' \in \partial(e)$ is designated an output. This parametrizes a map of vector spaces

$$\mathsf{V}[\mathbf{c}_1^{out};\mathbf{c}_1^{in}] \longrightarrow \mathsf{V}[\mathbf{c}_1^{out}\setminus\{a'\};\mathbf{c}_1^{in}\setminus\{a\}] = \mathsf{V}[\mathbf{c}_0^{out};\mathbf{c}_0^{in}]$$

which is a contraction operation $\xi^a_{a'}$ on the $\mathcal{S}^\uparrow$-bimodule. □

We can compose wiring diagrams of the type from Lemma 4.3 and Lemma 4.5 to get a new wiring diagram which parameterizes dioperadic composition. Let $H = (\{H_0, H_1, H_2\}, M, p)$ be a wiring diagram with $\partial(M)\cap H_1 \cap H_2 = \emptyset$ and $C = (\{C_0, C_1\}, N, q)$ be a wiring diagram which has a unique internal wire. Composition $C \circ H$, given by a bijection $H_0 \to C_1$,falls into one of three cases:

Case 1: The wiring diagram C identifies an input of H_1 with an output of H_1. In other words, the composite wiring diagram induces a map of vector spaces

$$\mathsf{V}[\mathbf{h}_1^{out};\mathbf{h}_1^{in}]\otimes\mathsf{V}[\mathbf{h}_2^{out};\mathbf{h}_2^{in}] \xrightarrow{H^*} \mathsf{V}[\mathbf{h}_1^{out},\mathbf{h}_2^{out};\mathbf{h}_1^{in},\mathbf{h}_2^{in}] \xrightarrow{C^*} \mathsf{V}[\mathbf{h}_1^{out}\setminus\{a'\},\mathbf{h}_2^{out};\mathbf{h}_1^{in}\setminus\{a\},\mathbf{h}_2^{in}].$$

This implies that the points a and a' are boundary points of the component of N forming the internal wire of C are also in H_1.

Case 2: The wiring diagram C identifies an input of H_2 with an output of H_2.

Case 3: The wiring diagram C identifies an element of H_1 with an element of H_2. This implies that the composite wiring diagram parameterizes a map of vector spaces

$$\mathsf{V}[\mathbf{h}_1^{out};\mathbf{h}_1^{in}]\otimes\mathsf{V}[\mathbf{h}_2^{out};\mathbf{h}_2^{in}]\xrightarrow{\circ_h}\mathsf{V}[\mathbf{h}_1^{out},\mathbf{h}_2^{out};\mathbf{h}_1^{in},\mathbf{h}_2^{in}]\longrightarrow\mathsf{V}[\mathbf{h}_1^{out}\setminus\{a'\},\mathbf{h}_2^{out};\mathbf{h}_1^{in},\mathbf{h}_2^{in}\setminus\{a\}].$$

Lemma 4.6. *The composite wiring diagrams described in Case* 1 *and Case* 2 *satisfy axiom* **??** *of Definition 3.3.*

Proof. The idea is that given H and C as above, we want to define a new composite wiring diagram $H'\circ C'$ which induces the same map of vector spaces as $C\circ H$. Moreover, we'd like that H' parameterizes a horizontal composition and C' parameterizes a contraction.

Case 1: Let H, C and $C\circ H$ be as above. Define a wiring diagram $C'=(\{C_0',C_1'\},N',q')$ with the property that N' has only one internal wire e and $\partial(e)\in C_1'$. Moreover, let $|C_1'|=|H_1|$ and define a bijection $i:H_1\to C_1'$ with the property that $q'(i(a))$ and $q'(i(a'))$ are in $\partial(N)\cap C_1'$. Now, define a wiring diagram $H'=(\{H_0',H_1',H_2'\},M',p')$ with $H_1'\cap H_2'\cap\partial(M')=\emptyset$, $|H_1'|=|C_0'|$ and $|H_2'|=|H_2|$. We can compose $H'\circ C'$ along a chosen bijection $j:H_1'\to C_0'$ and get the following diagram of linear maps.

$$\begin{tikzcd}
\mathsf{V}[\mathbf{h}_1^{out};\mathbf{h}_1^{in}]\otimes\mathsf{V}[\mathbf{h}_2^{out};\mathbf{h}_2^{in}] \arrow[r, "H^*"] \arrow[d, "i\otimes j"] & \mathsf{V}[\mathbf{h}_1^{out},\mathbf{h}_2^{out};\mathbf{h}_1^{in},\mathbf{h}_2^{in}] \arrow[r, "C^*"] & \mathsf{V}[\mathbf{h}_1^{out}\setminus\{a'\},\mathbf{h}_2^{out};\mathbf{h}_1^{in}\setminus\{a\},\mathbf{h}_2^{in}] \arrow[d] \\
\mathsf{V}[\mathbf{c'}_1^{out};\mathbf{c'}_1^{in}]\otimes\mathsf{V}[\mathbf{h}_2^{out};\mathbf{h}_2^{in}] \arrow[r, "C'^*\otimes id"] & \mathsf{V}[\mathbf{c'}^{out};\mathbf{c'}_0^{in}]\otimes\mathsf{V}[\mathbf{h}_2^{out};\mathbf{h}_2^{in}] \arrow[r, "H'^*"] & \mathsf{V}[\mathbf{c'}_0^{out},\mathbf{h}_2^{out};\mathbf{c'}_0^{in},\mathbf{h}_2^{in}]
\end{tikzcd}$$

The right-most map is clearly a vector space isomorphism and thus the composition wiring diagram $C\circ H$ is equivalent to $H'\circ C'$.

Case 2: The second case is identical to the first.

This implies that the horizontal and contraction operations satsify axiom **??** in Definition 3.3. □

We now turn our attention to the composite of wiring diagrams in Case 3.

Lemma 4.7. *Let $B=(\{B_0,B_1,B_2\},L,r)$ be a wiring diagram in which L is simply connected and L has precisely one component e with the property that $\partial(e)=\{a,b\}$, $a\in B_1$ and $b\in B_2$. Then B parametrizes a dioperadic composition of an $\mathcal{S}^{\uparrow}$-bimodule $\mathsf{V}=\{\mathsf{V}[m,n]\}$. Moreover, B is equivalent to a composite wiring diagram $C\circ H$ of type Case* 3

Proof. First, we may assume, without loss of generality that a is labelled as an output in B_1 and b is labelled as an input in $B2$. Our assumption that B has only one internal wire obtained by gluing these two endpoints amounts to giving a map of vector spaces

$$\mathsf{V}[\mathbf{b}_1^{out};\mathbf{b}_1^{in}]\otimes\mathsf{V}[\mathbf{b}_2^{out};\mathbf{b}_2^{in}]\longrightarrow\mathsf{V}[\mathbf{b}_1^{out}\setminus\{a\},\mathbf{b}_2^{out};\mathbf{b}_1^{in},\mathbf{b}_2^{in}\setminus\{b\}]=\mathsf{V}[\mathbf{b}_0^{out};\mathbf{b}_0^{in}]$$

which is a dioperadic operation ${}_a\circ_b$ on the $\mathcal{S}^{\uparrow}$-bimodule V.

For the second statement, let $H=(\{H_0,H_1,H_2\},M,p)$ be a wiring diagram with $H_1\cap H_2\cap\partial M=\emptyset$, $|H_1|=|B_1|$ and $|H_2|=|B_2|$ and let $C=(\{C_0,C_1\},N,q)$

be a contraction wiring diagram with the property that N has only one internal wire e with $\partial(e) = \{a, b\} \in C_1$. We define a parity preserving bijection $i : H_0 \to C_1$ with the property $i^{-1}(a) = p(h)$ for some $h \in H_1$ and $i^{-1}(b) = p(h')$ for some $h' \in H_2$. □

We can now state the main theorem of this section which is that every circuit algebra is a modular operad.

Theorem 4.8. *There is a functor* $L : \mathsf{CA} \to \mathsf{wProp}$*. This defines an equivalence of categories.*

Proof. Let V be a circuit algebra. For each pair of natural numbers m, n we have a vector space $\mathsf{V}[m; n]$. An element of this space is a chosen decoration of a wiring diagram $D = (\{D_0, D_1, \ldots, D_j\}), M, p)$ in which the outer disk D_0 has m output labels and n input labels.

We first claim that $\mathsf{V} = \{\mathsf{V}[m; n]\}_{m,n\geq 0}$ is an $\mathcal{S}^{\uparrow}$-bimodule. Let $\sigma = (\{S_0, S_1\}, M, \sigma)$ be the wiring diagram which consists of two disks S_0 and S_1 and a 1-manifold M with $|S_1| = |D_0|$ and $|S_1| = |S_0|$. Moreover, declare that the bijection from $D_0 \to S_1$ be the identity map. The wiring diagram σ parameterizes a map $\sigma : \mathsf{V}[m; n] \to \mathsf{V}[m; n]$ whose only possible action is to permute the ordering of the labels on D_0. Combining all such possible wiring diagrams we assemble the $\mathsf{V} = \{\mathsf{V}[m; n]\}_{m,n\geq 0}$ into an $\mathcal{S}^{\uparrow}$-bimodule.

Moreover, Lemma 4.3 and Lemma 4.5 show how one can construct the horizontal composition and contraction operations on the $\mathcal{S}^{\uparrow}$-bimodule V. Lemma 4.4 shows that the horizontal composition is unital, equivariant and associative. Lemma 4.6 shows that horizontal composition and contraction commute. The remainder of the axioms are similar and left to the reader. It follows that a circuit algebra V can be given the structure of a wheeled prop.

Let $f : \mathsf{V} \to \mathsf{W}$ be a map of circuit algebras. By definition, f commutes with all actions of wiring diagrams. It follows that f is a map of $\mathcal{S}^{\uparrow}$-bimodule which commutes with horizontal composition and contraction operations. This defines the functor L.

To show that the functor L is an equivalence of categories let $\{\mathsf{P}[m; n]\}_{m,n\geq 0}$ be the underlying $\mathcal{S}^{\uparrow}$-bimodule of a wheeled prop P. Elements of $\mathsf{P}[m; n]$ are decorations of an (m, n)-graph G (see, just above Definition 2.12 []). For a chosen decoration $w \in \mathsf{P}[m; n]$ it is straightforward to check that $w \simeq L\tilde{w}$ where $\tilde{w}$ is a decoration of a wiring diagram D in the shape of G . The image of w under the $\mathcal{S}_m \times \mathcal{S}_n$-action, $(\tau, \sigma) \cdot w$, is then isomorphic to $L\sigma\tilde{w}$ where σ is the wiring diagram giving the appropriate action on $\tilde{w}$. It follows that there exists a circuit algebra V so that $\mathsf{P} \simeq L\mathsf{V}$. Moreover, given an isomorphism of circuit algebras $i : \mathsf{V} \to \mathsf{W}$ $L(i)$ is an isomorphism of $\mathcal{S}^{\uparrow}$-bimodules and thus thus $L(i)$ is an isomorphism of wheeled props. It follows that L is an equivalence of categories.

□

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