

Automorphisms of seamed surfaces, modular operads and Galois actions

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A really fast introduction to a lot of cool math:

- Let $Gal(\mathbb{Q})$ denote the absolute Galois group of \mathbb{Q} .
- This is a large profinite group:

$$\widehat{G} = \varprojlim G/H$$

but we don't even know the finite quotients of $Gal(\mathbb{Q})$!

Idea : Identify $g \in Gal(\mathbb{Q})$ with a pair

$$(\chi(g), f_g) \in \widehat{\mathbb{Z}}^* \times \widehat{F}'_2$$

- $\chi(g)$ is the **cyclotomic character**.
- $\widehat{F}'_2 = \pi_1(\mathcal{M}_{0,4}) \cong \widehat{\Gamma}_{0,4}$.

A slightly easier group: \widehat{GT}

Notation: For any homomorphism of profinite groups

$$\widehat{F}_2 \longrightarrow G$$

$$(x, y) \longmapsto (a, b)$$

we write $f(a, b)$ for the image of any $f \in \widehat{F}_2$. For example:

- Given $id : \widehat{F}_2 \rightarrow \widehat{F}_2$, we have $f = f(x, y)$;
- Given the map $\widehat{F}_2 \rightarrow \widehat{F}_2$ which swaps generators x and y we have $f \mapsto f(y, x)$.

A slightly easier group: \widehat{GT}

The **Grothendieck-Teichmüller group** \widehat{GT} is the group of pairs

$$(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}'_2$$

satisfying the property that

$$x \mapsto x^\lambda \quad \text{and} \quad y \mapsto f^{-1}y^\lambda f$$

induce an automorphism of \widehat{F}_2 and :

- (I) $f(x, y)f(y, x) = 1$,
- (II) $f(x, y)x^m f(z, x)z^m f(y, z)y^m = 1$ where $xyz = 1$ and $m = (\lambda - 1)/2$,
- (III) $f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51})f(x_{12}, x_{23}) = 1$ in $\widehat{\Gamma}_{0,5}$
where x_{ij} is a Dehn twist of boundaries i and j .

Theorem (Ihara)

There is an injection $\text{Gal}(\mathbb{Q}) \hookrightarrow \widehat{GT}$.

So our question becomes

- What is \widehat{GT} ?

- ** Can we refine \widehat{GT} and thus get closer to $Gal(\mathbb{Q})$?

Goals:

Construct a **modular operad** S in groupoids with the following properties:

(1) Each $S(g, n)$ should “approximate” the mapping class group $\Gamma_{g,n}$, i.e.
 $BS(g, n) \simeq B\Gamma_{g,n}$

(2) The group $Gal(\mathbb{Q}) \subseteq \Lambda \subseteq \widehat{GT}$ defined by Nakamura-Schneps acts on the (profinite completion of the) modular operad S and

$$\text{End}_0(\widehat{S}) \cong \Lambda.$$

(3*) The action of the $Gal(\mathbb{Q})$ on $\widehat{BS} \simeq \widehat{\mathcal{M}}_{*,*}$ is non-trivial.

Modular Operads:

A **modular operad** P in a base symmetric monoidal category \mathcal{E} is:

(1) A bi-graded Σ -module $P = \{P(g, n)\}$ in \mathcal{E}

(2) associative, equivariant **compositions**

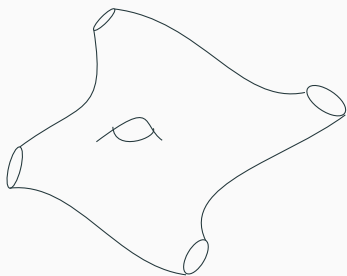
$$P(g, n) \times P(h, m) \xrightarrow{\circ_{ij}} P(g + h, n + m - 2)$$

(3) associative, equivariant **contractions**

$$P(g, n) \xrightarrow{\xi_{ij}} P(g + 1, n - 2)$$

Surfaces $\Sigma_{g,n}$.

Let $\Sigma_{g,n}$ denote a Riemann surface with n -boundaries:



The **mapping class group** of a surface Σ :

$$\Gamma_{g,n} = \pi_0 \text{Diff}^+(\Sigma, \partial).$$

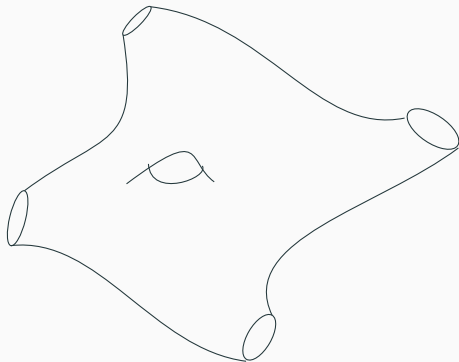
If you want to get ahead:

$$\widehat{\Gamma}_{g,n} \cong \pi_1(\mathcal{M}_{g,n}).$$

A nice fact about $\Gamma_{g,n}$

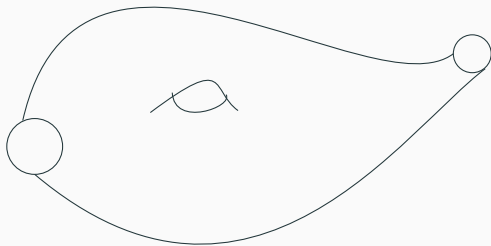
Hatcher-Thurston: The mapping class group has a presentation

$$\Gamma_{g,n} = \langle b_1, \dots, b_n, a_1, \dots, a_k \mid (C), (B), (D), (L) \rangle.$$



Pants Decompositions

A **pants decomposition** of $\Sigma_{g,n}$ is a collection of simple closed curves that cuts $\Sigma_{g,n}$ into *pairs of pants* (i.e. $\Sigma_{0,3}$).

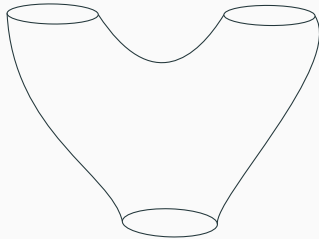
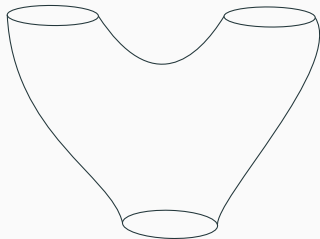


Quilted Pants Decompositions

We actually want **quilted** pants decompositions.

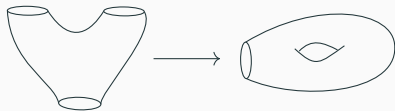
A **quilt** consists of:

- 2 marked points, called *vertices*, for every circle in the decomposition;
- 3 disjoint lines between the vertices for each pair of pants;



Quilted Pants Decompositions

A quilted **pants decomposition** can be determined by modular operadic composition.



A groupoid of quilted surfaces

The groupoid $S(g, n)$:

- objects are pairs (P, Q) giving an “atomic” quilt decomposition of $\Sigma_{g,n}$;
- morphisms are orientation preserving diffeomorphisms which fix the boundary labels (up to isotopy).

\Rightarrow

$$BS(g, n) \simeq B\Gamma_{g,n}.$$

A modular operad of Seamed Surfaces

Operations:

$$S_{g,n} \times_{ij} S_{h,k} \xrightarrow{\circ_{ij}} S_{g+h,n+k-2}$$

and

$$S_{g,n} \xrightarrow{\xi_{ij}} S_{g+1,n-2}$$

can be defined on objects by gluing surfaces and on morphisms as the "combination" of the maps on the subsurfaces.

These are well-defined, associative operations and thus

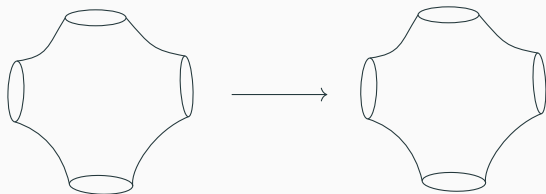
$$S = \{S_{g,n}\}$$

assembles into a **modular operad in groupoids**.

A groupoid of quilted surfaces

The morphisms in $S(g, n)$ are a composite of three types of elementary morphisms:

- A-moves



A groupoid of quilted surfaces

The morphisms in $S(g, n)$ are a composite of three types of elementary morphisms:

- S -moves



A modular operad of Seamed Surfaces

The elementary morphisms in $S(g, n)$ satisfy 6 important relations:

$$A_{\beta_3, \beta_4} A_{\beta_1, \beta_2} A_{\beta_4, \beta_5} A_{\beta_2, \beta_3} A_{\beta_5, \beta_1} = 1 \quad (5A)$$

$$S_{\beta_1, \beta_2} S_{\beta_2, \beta_3} S_{\beta_3, \beta_1} = 1 \quad (3S)$$

$$A_{\alpha_3, \epsilon_3} S_{\alpha_1, \alpha_2} A_{\epsilon_3, \epsilon_2} A_{\epsilon_2, \epsilon_1} S_{\alpha_2, \alpha_3} A_{\epsilon_1, \alpha_1} = 1. \quad (6AS)$$

The connection to \widehat{GT}

The genus 0 case

Proposition (Bovida-Horel-R)

There's an action of \widehat{GT} on the genus 0 part of our operad, \widehat{S}_0 .

Proposition (Bovida-Horel-R)

$$\widehat{GT} \cong \text{End}_0(\widehat{S}_0)$$

This uses in a fundamental way:

Proposition (Horel)

Let C and D be two groupoids (with finite sets of objects). The map

$$\widehat{C \times D} \rightarrow \widehat{C} \times \widehat{D}$$

is an isomorphism.

\Rightarrow The profinite completion $\widehat{S}_0 = \{\widehat{S(0, n)}\}$ is an operad in profinite groupoids.

Groups Related to \widehat{GT} : The higher genus case

Definition (Nakamura-Schneps)

Let Λ denote the group of pairs

$$(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}'_2$$

which satisfy the property that

$$x \mapsto x^\lambda \quad \text{and} \quad y \mapsto f^{-1}y^\lambda f$$

induce an automorphism of \widehat{F}_2 and relations (I) – (III) of \widehat{GT} and :

(IV)

$$f(e_1, a_1)a_3^{-8\rho_2}f(a_2^2, a_3^2)(a_3a_2a_3)^{2m}f(e_2, e_1)e_2^{2m}f(e_3, e_2) \\ a_2^{-2m}(a_1a_2a_1)^{2m}f(a_1^2, a_2^2)a_1^{8\rho_2}f(a_3, e_3) = 1$$

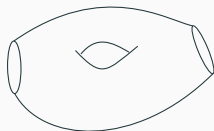
where a_1, a_2, a_3, e_1, e_2 are Dehn twists in $\Gamma_{1,2}$, $m = (\lambda - 1)/2$ and ρ_2 is an integer.

Groups Related to \widehat{GT} : The higher genus case

(IV)

$$f(e_1, a_1) a_3^{-8\rho_2} f(a_2^2, a_3^2) (a_3 a_2 a_3)^{2m} f(e_2, e_1) e_2^{2m} f(e_3, e_2) \\ a_2^{-2m} (a_1 a_2 a_1)^{2m} f(a_1^2, a_2^2) a_1^{8\rho_2} f(a_3, e_3) = 1$$

where a_1, a_2, a_3, e_1, e_2 are Dehn twists in $\Gamma_{1,2}$, $m = (\lambda - 1)/2$ and ρ_2 is an integer.



Nakamura-Schneps: $Gal(\mathbb{Q}) \subseteq \Lambda \subseteq \widehat{GT}$.

Groups Related to \widehat{GT} : The higher genus case

Horel's result applies to the modular operad $S = \{S(g, n)\}$ and

$$\widehat{S} = \{\widehat{S}(g, n)\}$$

is a modular operad in profinite groupoids.

Proposition (Bonatto-R)

There is an action

$$\Lambda \longrightarrow \text{End}_0(\widehat{S}).$$

Given a pair $(\lambda, f) \in \Lambda$:

$$(\lambda, f) \longrightarrow \begin{cases} a_{\frac{1}{2}} \mapsto a_{\frac{1}{2}}^\lambda \\ A_{\alpha, \beta} \mapsto A_{\alpha, \beta} \cdot f(a, b) a^{N(\lambda-1)/2} \\ S_{\alpha, \beta} \mapsto S_{\alpha, \beta} \cdot (aba)^{\lambda-1} b^{N(\lambda-1)/2-8\rho_2} f(a^2, b^2) a^{8\rho_2} \end{cases}$$

where N is some integer (a “twist”) and ρ_2 is a different integer.

Is this action interesting?

The spaces $BS(0, n)$ are very closely related to the moduli space $\bar{\mathcal{M}}_{0,n}$:

Theorem (Drummond-Cole)

There's a homotopy pushout of operads :

$$\begin{array}{ccc} S^1 & \longrightarrow & BS(0, n) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \bar{\mathcal{M}}_{0,n} \end{array}$$

The idea:

- Use $\widehat{GT} \cong \text{End}_0(\widehat{S}_0) = \pi_0(\mathbb{R} \text{End}(\widehat{BS}_0))$ to get an action of \widehat{GT} (and thus $\text{Gal}(\mathbb{Q})$) on $BS(0, n)$
- In this case, profinite completion interacts well with pushouts of ∞ -operads \Rightarrow the Galois action is non-trivial.

Groups Related to \widehat{GT} : What do we know in the higher genus case ?

Recall:

- (1) $Gal(\mathbb{Q}) \subseteq \Lambda \subseteq \widehat{GT}$
- (2) The spaces $BS(g, n) \simeq B\Gamma_{g,n}$ are closely related to the spaces $\bar{\mathcal{M}}_{g,n}$

To study the action of $Gal(\mathbb{Q})$ on the modular operad \widehat{BS} :

- We no longer have the ability to compare with ribbon braids in genus 0;
- The classifying spaces $BS_{g,n}$ are not “nice” for $g \geq 2$ and so the best you can do:

Theorem (Bonatto-R)

$$\pi_0 \mathbb{R}End(\widehat{NBS}) \simeq \pi_0 \mathbb{R}End(\widehat{NBS}_{\leq 1})$$