Lecture 1: Graphs and Modular Operads

Cyclic Operads

A $\mathfrak{C}$-coloured **cyclic operad** is an algebraic structure consisting of:

- an involutive set of colours $\mathfrak{C}$;
- for each $c_1, \ldots, c_n \in \mathfrak{C}$ a Σ_n-set $\mathsf{P}(c_1, \ldots, c_n)$;
- a family of equivariant, associative and unital composition operations

$$\mathsf{P}(c_1, \ldots, c_n) \times \mathsf{P}(d_1, \ldots, d_m) \longrightarrow \mathsf{P}(c_1, \ldots, \hat{c}_i, \ldots, d_1, \ldots \hat{d}_j, \ldots, d_m),$$

when $c_i = d_j$.

Modular Operads

A $\mathfrak{C}$-coloured **modular operad** is a cyclic operad which also has

- a family of equivariant contraction operations

$$\mathsf{P}(c_1, \ldots, c_n) \longrightarrow \mathsf{P}(c_1, \ldots, \hat{c}_i, \ldots, \hat{c}_j, \ldots c_n),$$

  when $c_i = c_j$.

satisfying some axioms.

Example: A modular operad of surfaces

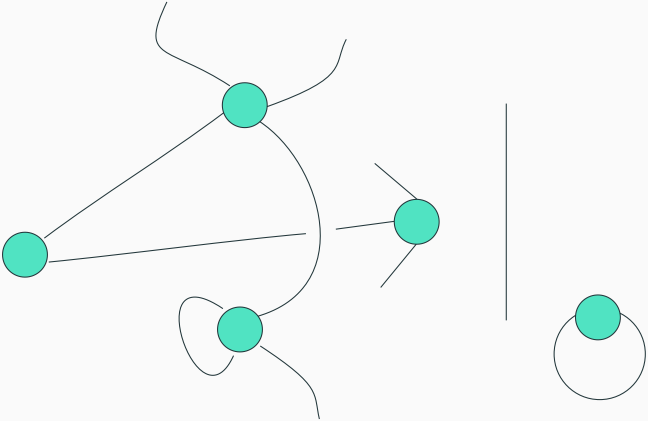
Exercise: $*$-autonomous categories

$*$-autonomous categories are closed symmetric monoidal categories with a global dualizing object so that $(a^\dagger)^\dagger \cong a$. Show that all strict $*$-autonomous categories are examples of cyclic operads. (See Example 2.2 in D-C–H).

Goal for Today:

There is an equivalence of categories:

$$\mathrm{ModOp} \xrightarrow[\cong]{N} \mathbf{Set}^{\mathsf{U}^{op}}_{Segal}.$$

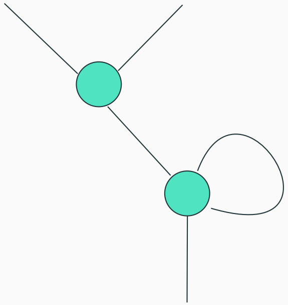


Graphs:

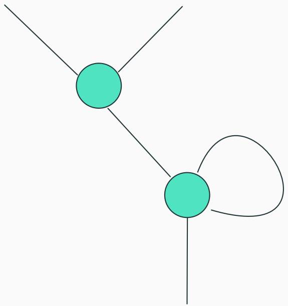
A **graph** G is a diagram of finite sets:

$$i \circlearrowright A \xleftarrow{s} D \xrightarrow{t} V$$

- i is a free involution;
- s is a monomorphism.



Graphs: Edges and Internal Edges



The involution on half-edges $i : a \mapsto a^\dagger$ determines the **edges** of a graph.

- An **edge** is just an i-orbit $[a, a^\dagger]$.
- An **internal edge** is an edge of the form $[b, b^\dagger]$ where both b and $b^\dagger$ are in D.

Special Graphs, Boundaries and Neighbourhoods

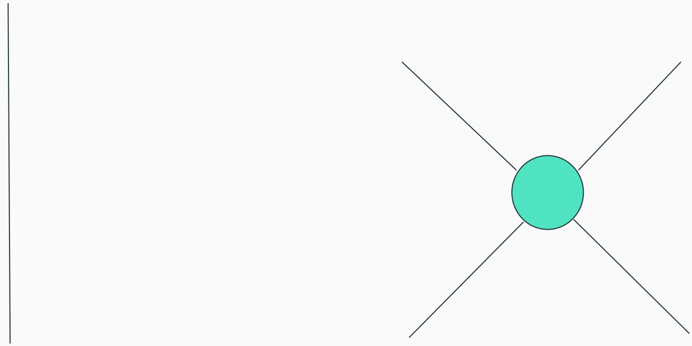
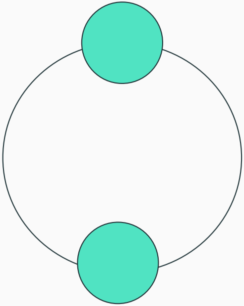


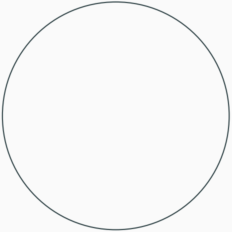
Figure 1: The exceptional edge $\updownarrow$ and the 4-star $\star_4$.

Definition

- The **boundary** of a graph: $\partial(G) = A \setminus D$.
- The **neighbourhood** of $v \in V(G)$: $\mathrm{nb}(v) = t^{-1}(v) \subseteq D$.

A loop with 2 nodes and a nodeless loop





Exercise

Draw a graph G with $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $D = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $V = \{v_1, v_2, v_3, v_4\}$ and $i(n) = n - 1$ for $n = 2, 4, 6, 8$.

Exercise: The star of a vertex and star of a graph

Definition

- The **star of a graph** $\bigstar_G$ is the one-vertex graph with $A = \partial(G) \sqcup \partial(G)^\dagger$, $D = \partial(G)^\dagger$ and $\partial(\bigstar_G) = A \setminus D = \partial(G)$. In other words, $\bigstar_G = \bigstar_{\partial(G)}$.
- If v is a vertex in G, the **star of a vertex** $\bigstar_v$ is the one-vertex graph with $A = \text{nb}(v) \sqcup \text{nb}(v)^\dagger$ and $\partial(\bigstar_v) = \text{nb}(v)^\dagger$.

Exercise
For the graph G drawn above, write down $\bigstar_G$ and $\bigstar_v$ for each $v \in V(G)$.

Morphisms of Graphs

Graphs are diagrams in FinSet in the shape of

$$\mathcal{I} := \quad i \circlearrowright \bullet \xleftarrow{s} \bullet \xrightarrow{t} \bullet$$

Definition
A natural transformation $f : G \to G'$ is called an **embedding** if

$$\begin{array}{ccccc} i \circlearrowright A & \xleftarrow{s} & D & \xrightarrow{t} & V \\ \downarrow f & & \downarrow f & & \downarrow f \\ i' \circlearrowright A' & \xleftarrow{s'} & D' & \xrightarrow{t'} & V' \end{array}$$

- the right-hand square is a pullback;
- $V \to V'$ a monomorphism.

There is a class of **vertex embeddings**

$$\bigstar_v \hookrightarrow G$$

for every $v \in V(G)$:

Example

There is a class of **vertex embeddings**

$$\bigstar_v \hookrightarrow G$$

for every $v \in V(G)$:

Example

Explicitly:

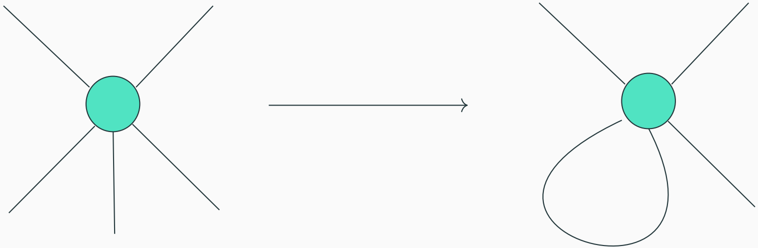
$$\begin{array}{ccccc} \mathrm{nb}(v) \sqcup \mathrm{nb}(v)^{\dagger} & \xleftarrow{s} & \mathrm{nb}(v) & \xrightarrow{t} & \{v\} \\ \downarrow & & \downarrow & & \downarrow \\ A & \xleftarrow{s'} & D & \xrightarrow{t} & V. \end{array}$$

Embeddings are not necessarily injective on half-edges.

There is a natural embedding

$$\bigstar_n \hookrightarrow \xi\bigstar_n$$

which is not injective on half-edges.



Definition

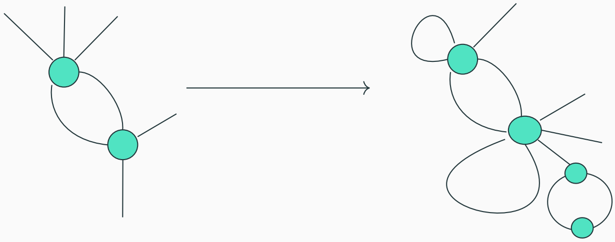
A graphical map $\varphi : G \to G'$ consists of:

- a map of involutive sets $\varphi_0 : A \to A'$;
- a function $\varphi_1 : V \to \mathrm{Emb}(G')$ satisfying the following conditions:
  - The embeddings $\varphi_1(v)$ have no **overlapping** vertices
  - For each v, the diagram:

$$\begin{array}{ccc} \mathrm{nb}(v) & \xrightarrow{\;i\;} & A \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\varphi_0} \\ \partial(\varphi_1(v)) & \longrightarrow & A' \end{array}$$

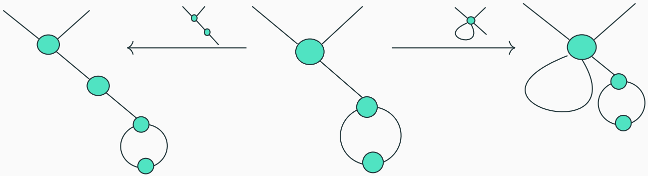
    commutes.
  - If $\partial(G) = \emptyset$, then there exists a v in V so that $\varphi_1(v) \neq \updownarrow$.

Graphical Maps



Inner coface maps

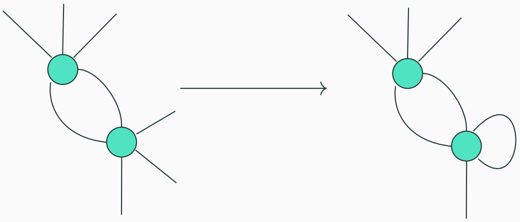
An **inner coface map** $d_v : G \to G'$ is a graphical map defined by "blowing-up" a single vertex v in G by a graph $(d_v)_1$ which has precisely **one** internal edge.



Outer coface maps

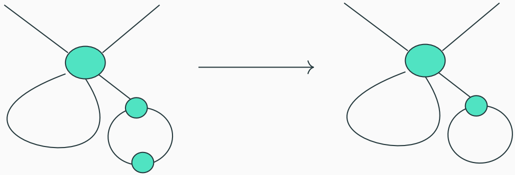
An **outer coface map** is either:

- an **embedding** $d_e : G \to G'$ in which G' has precisely **one** more internal edge than G or
- an **embedding** $\updownarrow \to \bigstar_n$.



Codegeneracy maps

A **codegeneracy map** $s_v : G \to G'$ is a graphical map defined by "blowing-up" a vertex v in G by $\updownarrow$.

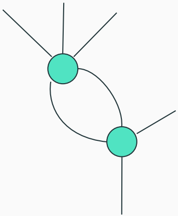


The graphical category

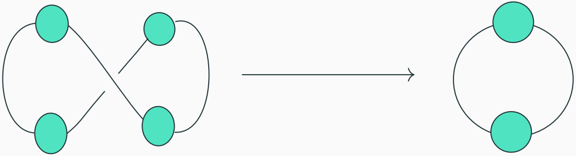
The **graphical category** U is the category whose objects are connected graphs. The morphisms are the graphical maps.

The modular operad $\langle G \rangle$ generated by a graph G is the free modular operad whose:

- set of colours is the set of half-edges A;
- a collection $E(a_1, \ldots, a_n) = \begin{cases} \{v\} \text{ if } (a_1, \ldots, a_n) = \partial(\star_v) \\ \emptyset \text{ otherwise.} \end{cases}$
- $\langle G \rangle = F(E)$



Proposition (Proposition 2.25 HRY2)
The assignment $G \mapsto \langle G \rangle$ *defines a faithful functor* $\mathsf{U} \to \mathsf{ModOp}$ *which is injective on isomorphism classes of objects.*



Graphical Sets

The category of **graphical sets** is $\mathsf{Set}^{\mathsf{U}^{op}}$.

- X_G: evaluation of X at $G \in \mathsf{U}$.
- $\varphi : G \to G' \Rightarrow \varphi^* : X_{G'} \to X_G$.
- The **representable presheaf** at G:

$$\mathsf{U}[G] := \mathsf{U}(-; G)$$

$$\mathsf{U}[G]_H := \mathsf{U}(H, G)$$

for all graphs H.

Segal Maps

Internal edges $\Rightarrow$ diagram of embeddings

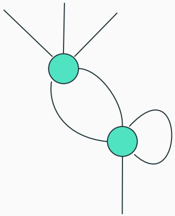
$$\bigstar_v \longleftarrow \updownarrow \longrightarrow \bigstar_w$$

in U.

Let

$$X_G^1 = \lim_{\bigstar_v \leftarrow \updownarrow \rightarrow \bigstar_w} \left(\begin{array}{ccc} X_{\bigstar_v} & & X_{\bigstar_w} \\ & \searrow \quad \swarrow & \\ & X_{\updownarrow} & \end{array} \right).$$

Segal Maps



Segal Maps

The embeddings $\bigstar_v \hookrightarrow G$ induce a Segal map

$$X_G \longrightarrow X_G^1 \subseteq \prod_{v \in V(G)} X_{\bigstar_v}$$

which factors through X_G^1.

Segal Maps

The embeddings $\bigstar_v \hookrightarrow G$ induce a Segal map

$$X_G \longrightarrow X^1_G \subseteq \prod_{v \in V(G)} X_{\bigstar_v}$$

which factors through X^1_G.

Exercise
In the case when $X_{\updownarrow} = *$, show that $X^1_G = \prod_{v \in V(G)} X_{\bigstar_v}$.

Segal graphical sets

A graphical set $X \in \mathsf{Set}^{\mathsf{U}^{op}}$ is strictly **Segal** if the Segal map

$$X_G \longrightarrow X^1_G \subseteq \prod_{v \in V(G)} X_{\bigstar_v}$$

is a bijection for each G in U.

The Nerve

$$N : \mathsf{ModOp} \longrightarrow \mathsf{Set}^{\mathsf{U}^{op}}$$

$$N\mathsf{P}_G = \mathsf{ModOp}(\langle G \rangle, \mathsf{P})$$

for any $\mathsf{P} \in \mathsf{ModOp}$ and any $G \in \mathsf{U}$.

The Nerve

$$N : \mathsf{ModOp} \longrightarrow \mathsf{Set}^{\mathsf{U}^{op}}$$

$$N\mathsf{P}_G = \mathsf{ModOp}(\langle G \rangle, \mathsf{P})$$

for any $\mathsf{P} \in \mathsf{ModOp}$ and any $G \in \mathsf{U}$.

$N\mathsf{P}_G$ is "the set of P decorations of the graph G":

- $N\mathsf{P}_{\updownarrow} = \mathsf{ModOp}(\langle \updownarrow \rangle, \mathsf{P}) = \mathfrak{C}$;
- $N\mathsf{P}_{\star_n} = \mathsf{ModOp}(\langle \star_n \rangle, \mathsf{P}) = \mathsf{P}(c_1, \dots c_n)$.

The Nerve Theorem

Theorem (Theorem 3.6 HRY2)
The nerve functor is fully faithful. Moreover, the following statements are equivalent for $X \in \mathsf{Set}^{\mathsf{U}^{op}}$.

- *There exists a modular operad P and an isomorphism $X \cong N\mathsf{P}$.*
- *X satisfies the strict Segal condition.*

Theorem (Theorem 3.6 HRY2)
The nerve functor is fully faithful. Moreover, the following statements are equivalent for $X \in \mathsf{Set}^{\mathsf{U}^{op}}$.

- *There exists a modular operad* P *and an isomorphism* $X \cong N\mathsf{P}$.
- X *satisfies the strict Segal condition.*

In other words:

$$\mathsf{ModOp} \xrightarrow[\cong]{N} \mathbf{Set}^{\mathsf{U}^{op}}_{Segal}.$$

Exercise: Graphs exhibit some strange behaviour

Given a graph $G \in \mathsf{U}$, we now have two ways to assign an object in $\mathsf{Set}^{\mathsf{U}^{op}}$ to G:

- the representable presheaf $\mathsf{U}[G]$,
- taking the nerve of the modular operad $\langle G \rangle$, $N \langle G \rangle$.

The representable $\mathsf{U}[G]$ is a sub-object of $N \langle G \rangle$ (since $J : \mathsf{U} \to \mathsf{ModOp}$ is faithful) but they nearly never coincide.

- Let G be the loop with one node and show $\mathsf{U}[G] \subset N \langle G \rangle$.
- Show that we have $\mathsf{U}[\star_0] = N \langle \star_0 \rangle$.

Further Directions

Earlier we defined the notion of (inner and outer) coface maps of U. Given a coface map δ with codomain G, one can define the **horn** $\Lambda^{\delta}[G]$ which is a sub-object of the representable object $\mathsf{U}[G]$. A **strict inner Kan** graphical set is a presheaf $X \in \mathsf{Sets}^{\mathsf{U}^{op}}$ such that every diagram

$$\begin{array}{ccc} \Lambda^{\delta}[G] & \longrightarrow & X \\ \downarrow & \nearrow & \\ \mathsf{U}[G] & & \end{array}$$

with δ an inner coface map admits a unique filler.

Further Directions

Michelle Strumila shows in her PhD thesis that :

Theorem (Strumila)
The nerve functor

$$N : \mathsf{ModOp} \longrightarrow \mathsf{Set}^{\mathsf{U}^{op}}$$

is fully faithful. Moreover, the following statements are equivalent for $X \in \mathsf{Set}^{\mathsf{U}^{op}}$.

There exists a modular operad P *and an isomorphism* $X \cong N\mathsf{P}$.

X *satisfies the strict Segal condition.*

X *is strict inner Kan.*

If one relaxes the inner Kan condition you arrive at a model for **quasi** or ∞-**modular operads**. Following the example of dendroidal sets, one could find a model category structure in which the weak inner Kan graphical sets are the fibrant objects. Because the graphical categories for cyclic operads, wheeled properads, etc can all be derived from U this would simultaneously create models for many flavours of ∞-"operads".