A topological characterisation of the Kashiwara-Vergne groups

Marcy Robertson (University of Melbourne)

joint with: Zsuzsi Dancso (Sydney) and Iva Halacheva (Northeastern)

Universal Finite Type Invariants or "Expansions":

class of knotted objects (eg: tangles)

$$\mathcal{K} \xrightarrow{Z} \mathrm{A}$$

parenthesized braids

graded vector space + structure

- Bar-Natan : When $\mathcal{K} = \mathrm{PaB}$ and $\mathrm{A} = \mathrm{PaCD}$,

parenthesized chord diagrams

$$\{Z\} \rightleftarrows \{\text{Associators}\}.$$

- Drinfeld : There exists a pair of prounipotent groups $\widehat{\mathrm{GT}}$ and GRT

$$\widehat{\mathrm{GT}} \quad \circlearrowright \{\mathcal{K} \xrightarrow{Z} \mathrm{A}\} \circlearrowleft \quad \mathrm{GRT}$$

- Bar-Natan, Fresse,... : There exist isomorphisms

$$\widehat{\mathrm{GT}} \cong \mathrm{Aut}_0(\widehat{\mathrm{PaB}}) \quad \text{and} \quad \mathrm{GRT} \cong \mathrm{Aut}_0(\mathrm{PaCD}).$$

completed operad

Expansions and The Kashiwara-Vergne Conjecture:

- Bar-Natan and Dancso: Homomorphic Expansions of *w*-**foams**

$$\mathrm{wF} \xrightarrow{Z} \mathrm{A}$$

  are in 1-to-1 correspondence with solutions to the Kashiwara-Vergne Conjecture.

- Alekseev-Torossion; Alekseev-Enriques-Torossion : There exists a pair of prounipotent groups KV and KRV

$$\mathrm{KV} \quad \circlearrowright \mathrm{wF} \xrightarrow{Z} \mathrm{A} \circlearrowleft \quad \mathrm{KRV}$$

- Dancso-Halacheva-R : There exist isomorphisms

$$\mathrm{KV} \cong \mathrm{Aut}_v(\widehat{\mathrm{wF}}) \quad \text{and} \quad \mathrm{KRV} \cong \mathrm{Aut}_v(\mathrm{A}).$$

The Kashiwara-Vergne Conjecture:

- Alekseev-Torossion; Alekseev-Enriques-Torossion : The groups KV and KRV are closely related to $\widehat{\mathrm{GT}}_1$ and GRT_1:

$$\begin{tikzcd}
\mathrm{KV} & \mathrm{wF} \arrow[loop left] \arrow[rr, "Z"] & & \mathrm{A} \arrow[loop right] & \mathrm{KRV} \\
\widehat{\mathrm{GT}}_1 \arrow[u] & \mathrm{PaB} \arrow[loop left] \arrow[u, dotted] \arrow[rr] & & \mathrm{PaCD} \arrow[loop right] \arrow[u, dotted] & \mathrm{GRT}_1 \arrow[u, "\checkmark"']
\end{tikzcd}$$

- Dancso-Halacheva-R: There exists an isomorphism

a bubble condition

$$\mathrm{GRT}_1 \cong \mathrm{Aut}_v^b(\mathrm{A}).$$

Circuit Algebras:

$\mathcal{K} = wF$

"non-planar planar algebra"

Definition: A **circuit algebra** is an algebra over the operad of **wiring diagrams**.

Theorem (Dancso-Halacheva-R)
There is an equivalence of categories:

$$\{\text{oriented circuit algebras}\} \cong \{\text{wheeled props}\}$$

Wiring Diagrams:

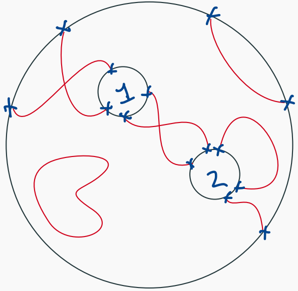
A **wiring diagram** consists of:

(1) A disc with n-holes: $D_0 \setminus \{\mathring{D}_1 \cup \ldots \mathring{D}_n\}$

(2) A set of distinguished (input and output) labels L_k on ∂D_k, $0 \leq k \leq n$.

(3) An immersed (oriented) 1-manifold M together with a bijection

$$\partial M \rightleftarrows \bigcup_k \cancel{\partial D_k}\, L_k$$

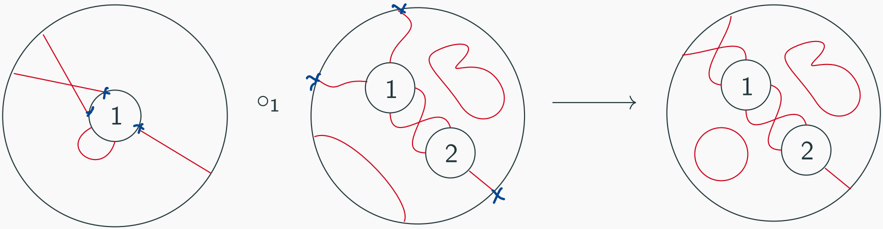


* independant of specific immersion

Wiring Diagrams:

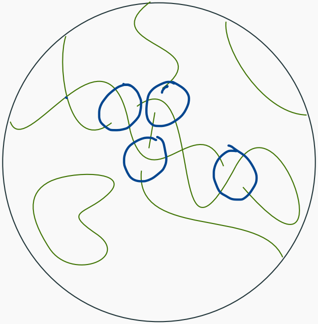
$\mathbb{N} \times \mathbb{N}$

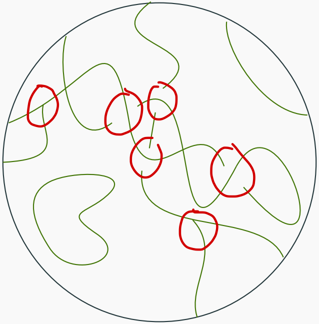
The collection of all wiring diagrams forms a **coloured operad** WD



The circuit algebra of w-foams

- ***w*-tangles** are embeddings of (framed) surfaces into $\mathbb{R}^4$ up to ambient isotopy.
- ***w*-foams** are w-tangles where we allow "foamed vertices" – intersections of two tubes which may "merge" or "split."





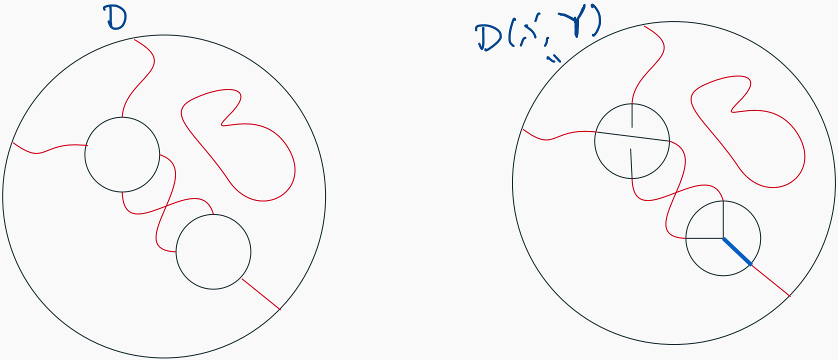
The circuit algebra of w- **foams** is:

$$\mathrm{wF} = \mathrm{CA}\left\langle \times, \times, \curlywedge, \uparrow \mid R1^s, R2, R3, R4, OC, CP \right\rangle .$$

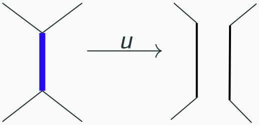
strand deletion

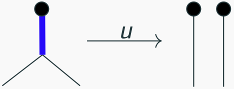
Moreover, wF is equipped with auxiliary operations: S_e, A_e, d_e and **unzips** u_e.

orientation switches



The circuit algebra of w-foams



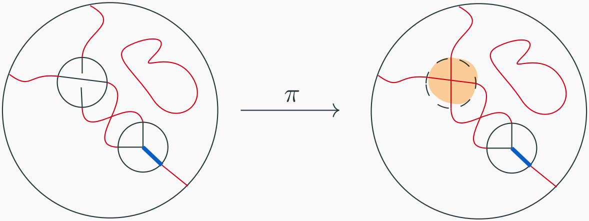


The w-foam skeleton

The circuit algebra of *w*-foam **skeleta** is the circuit algebra

$$\mathcal{S} = \mathrm{CA}\left\langle \;,\; \right\rangle$$

with auxiliary operations: S_e, A_e, u_e and d_e.



$$\mathrm{wF} := \coprod_{s\in\mathcal{S}} \pi^{-1}(s) = \coprod_{s\in\mathcal{S}} \mathrm{wF}(s).$$

Let $\mathbb{Q}[\mathrm{wF}](s)$ denote the $\mathbb{Q}$-vector space of formal linear combinations of w-foams T_i with skeleton $s \in \mathcal{S}$,

$$\sum_{T_i \in \mathrm{wF}(s)} \alpha_i T_i \in \mathbb{Q}[\mathrm{wF}](s).$$

$D(p_1, \ldots, p_k) \in \mathcal{I}$ if one of the p_j is in $\mathcal{I}$

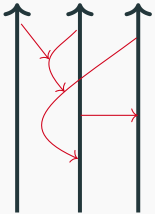
$\mathcal{I}^m$ if at least m of the p_j are in $\mathcal{I}$

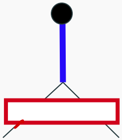
The (prounipotent) **completion** $\widehat{\mathrm{wF}} = \coprod_{s \in \mathcal{S}} \widehat{\mathrm{wF}}(s)$ where

$$\widehat{\mathrm{wF}}(s) := \lim \left(\mathbb{Q}[\mathrm{wF}](s)/\mathcal{I}(s) \leftarrow \mathbb{Q}[\mathrm{wF}](s)/\mathcal{I}^2(s) \leftarrow \mathbb{Q}[\mathrm{wF}](s)/\mathcal{I}^3(s) \leftarrow \ldots \right)$$

The associated graded: arrow diagrams

Arrow diagrams are oriented Jacobi diagrams on *w*-foam skeleton.



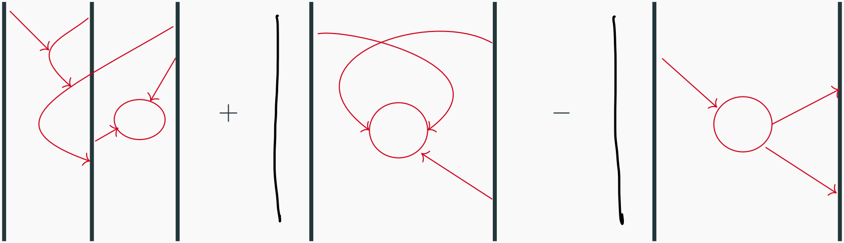


The associated graded: arrow diagrams

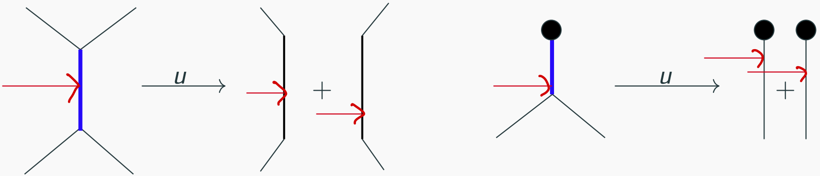
The circuit algebra of **arrow diagrams** is the complete circuit algebra in $\mathbb{Q}$-vector spaces

$$\mathsf{A} := \mathsf{CA}\left\langle \uparrow\!\!\mapsto\!\!\uparrow,\ \uparrow,\ \curlywedge \;\middle|\; \mathsf{4T},\ \mathsf{TC},\ \mathsf{VI},\ \mathsf{CP},\ \mathsf{RI} \right\rangle$$

with the associated graded of the w-foam skeleton operations: S_e, A_e, d_e, unzips u_e.



The associated graded: arrow diagrams



How does this relate to the KV solutions?

degree complete

A **tangential derivation** on $\widehat{\mathfrak{lie}}_n$ is a derivation u of $\widehat{\mathfrak{lie}}_n$ which acts on the generators x_i by $u(x_i) = [x_i, a_i]$, for some $a_i \in \widehat{\mathfrak{lie}}_n$.

$u = (a_1, a_2)$ where $a_i \in \mathfrak{lie}_2$

$u(x) = [x, a_1]$
$u(y) = [y, a_2]$

$TAut_2 = \exp(tder_2)$

Precisely:

$$\mathsf{SolKV} = \mathsf{SolKV}(\mathbb{Q}) := \Big\{ (F, r) \in \mathsf{TAut}_2(\mathbb{Q}) \times u^2\mathbb{Q}[[u]] \,\Big|\, F(e^x e^y) = e^{x+y} \text{ and } J(F) = \mathrm{tr}\Big(r(x+y) - r(x) - r(y) \Big) \Big\}.$$

Jacobian

How does this relate to the KV solutions?

Ex (1) $t^{1,2} = (y, x) \in \mathfrak{tder}_2$.

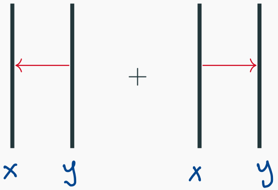
Ex (2) $u = (0, [[x, y], z], 0) \in \mathfrak{tder}_3$

$t^{1,2}(x) = [x,y]$

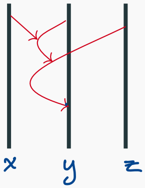
$t^{1,2}(y) = [y,x]$

root determines this position in the tuple

(y,x)



$[[x,y],z]$



How does this relate to the KV solutions?

KV solutions live in $U(\mathfrak{tder}_n \ltimes \mathrm{cyc}_n)$

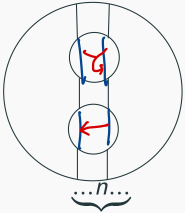
TAut

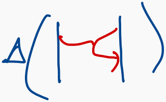
There is an isomorphism of Hopf algebras

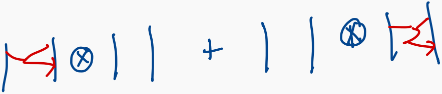
$$\mathrm{A}(\uparrow_n) \cong \hat{U}((\mathfrak{tder}_n \oplus \cancel{\mathfrak{a}_n}) \ltimes \mathrm{cyc}_n).$$

for the Jacobian part

"wheels







same coproduct as chord diagrams

Expansions

A **homomorphic expansion** of w-foams is a circuit algebra homomorphism $Z : \mathsf{wF} \to \mathsf{A}$ such that:

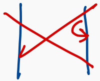
- its linear extension $Z : \mathbb{Q}[\mathsf{wF}] \to \mathsf{A}$ is a filtered, skeleton preserving homomorphism
- $\mathrm{gr} Z = \mathrm{id}_{\mathsf{A}}$.

Theorem (Dancso-Halacheva-R)
For any homomorphic expansion the induced map $\widehat{Z} : \widehat{\mathsf{wF}} \to \mathsf{A}$ is an isomorphism.

Theorem (Bar-Natan – Dancso)
There is a one-to-one correspondence:

$$\{\text{v-small exp}\} \rightleftarrows \{\text{SolKV}\}.$$

v-small
= kills off that
a_n



The graded group KRV

uniquely determined by α

$$\mathrm{KRV} = \mathrm{KRV}(\mathbb{Q}) := \Big\{ (\alpha, s) \in \mathrm{TAut}_2(\mathbb{Q}) \times u^2\mathbb{Q}[[u]] \,\Big|\, \alpha(e^{x+y}) = e^{x+y} \text{ and } J(\alpha) = \mathrm{tr}\Big(s(x+y) - s(x) - s(y)\Big) \Big\}$$

SolKV $\circlearrowleft$ KRV is given by $F \cdot \alpha = \alpha^{-1} \circ F$.

Theorem (Dancso-Halacheva-R)
There is an isomorphism of groups $\mathrm{Aut}_v(\mathrm{A}) \cong \mathrm{KRV}$.

The group KV

$$\mathsf{KV} = \mathsf{KV}(\mathbb{Q}) := \Big\{ (a,\sigma) \in \mathsf{TAut}_2(\mathbb{Q}) \times u^2\mathbb{Q}[[u]] \,\Big|\, a(e^x e^y) = e^x e^y \text{ and } J(a) = \mathrm{tr}\Big(\sigma(\mathfrak{bch}(x,y)) - \sigma(x) - \sigma(y)\Big) \Big\}$$

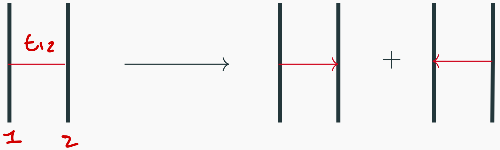
$\mathsf{KV} \circlearrowright \mathsf{SolKV}$ is given by $a \cdot F = F \circ a^{-1}$.

Theorem (Dancso-Halacheva-R)
There is an isomorphism of groups $\mathsf{Aut}_v(\widehat{\mathsf{wF}}) \cong \mathsf{KV}$.

- **chord diagrams** are pictorial depictions of $\hat{U}(\mathfrak{t}_n)$
- There's an inclusion of Lie algebras $\mathfrak{t}_n \hookrightarrow \mathfrak{tder}_n$.

Lie alg generated by t_{ij}

$t^{i,j} = (0, \ldots, \underset{i}{y}, \ldots, \underset{j}{x}, \ldots 0)$



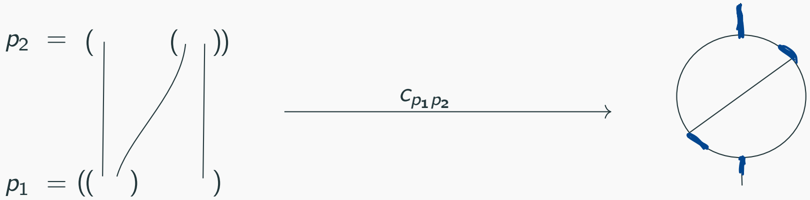
Lemma (Dancso-Halacheva-R)
For each $n \geq 2$ there exists an inclusion of Hopf algebras

$$\mathrm{Hom}_{\mathrm{CD}(n)}(*, *) \xrightarrow{\varepsilon} \mathsf{A}(\uparrow_n).$$

The relationship between KRV and GRT_1

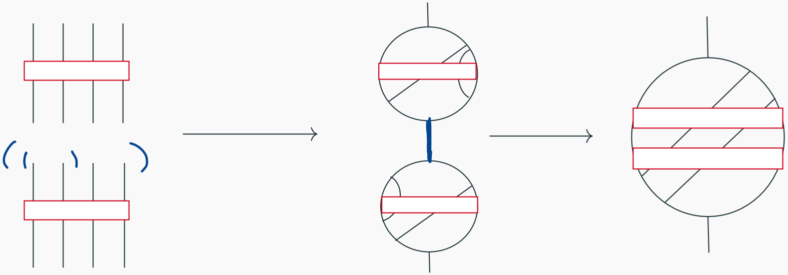
Theorem (Bar-Natan, Fresse)
There is an isomorphism of groups $\mathrm{Aut}_0(\mathrm{PaCD}) \cong \mathrm{GRT}_1$.

PaCD(3)

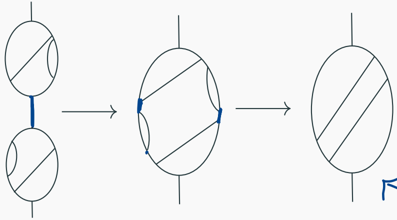


Theorem (Dancso-Halacheva-R)
There is an inclusion of Hopf groupoids

$$\mathrm{PaCD}(n) \xrightarrow{\varepsilon} \bigcup_{c_{p_i p_j}} \mathrm{A}(c_{p_i p_j}).$$

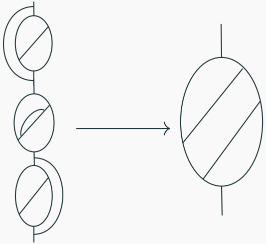


The pentagon



bubble

$$=$$



Theorem (Dancso-Halacheva-R)

The (isomorphic) image of GRT_1 *in* $\mathrm{Aut}_v(\mathrm{A})$ *is the subgroup*

$\mathrm{E} = \{\tilde{G} \in \mathrm{Aut}_v(\mathrm{A}) : \tilde{G}(\oslash) = \varepsilon(G(|\!/\!|))$ *for some (unique)* $G \in \mathrm{Aut}_0(\mathrm{PaCD})\}$.

Thanks!