# A topological characterisation of the Kashiwara-Vergne groups

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# Universal Finite Type Invariants or "Expansions":

$$\mathcal{K} \xrightarrow{Z} \mathsf{A}$$

- Bar Natan and Dancso: Homomorphic Expansions of $w$-**foams**

$$\mathsf{wF} \xrightarrow{Z} \mathsf{A}$$

are in 1-to-1 correspondence with solutions to the Kashiwara-Vergne Conjecture.

# Expansions and The Kashiwara-Vergne Conjecture:

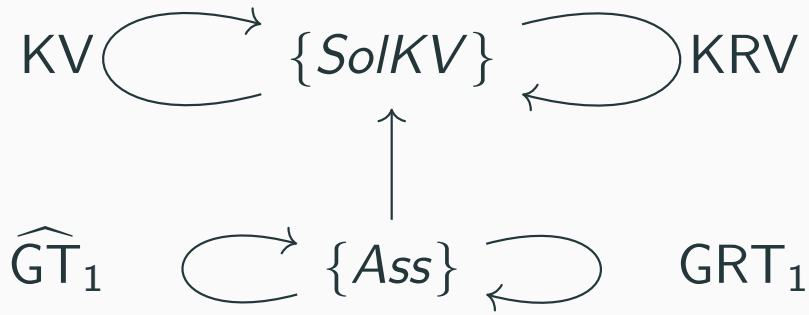
- If $xy = yx$, we know that $e^x e^y = e^{x+y}$.
- In $xy \neq yx$ then

$$e^x e^y = e^{x+y+\frac{1}{2}[x,y]+\frac{1}{12}[x,[x,y]]+\ldots},$$

  where $[x, y] = xy - yx$.
- The KV-conjecture asks: When can we write $e^x e^y$ in terms of convergent power series and adjoint operations?

# The Kashiwara-Vergne Conjecture:

- Alekseev-Torossion; Alekseev-Enriques-Torossion : Solutions to the KV conjecture are closely related to quantised deformation theory.




- Drinfeld : There exists a pair of prounipotent groups $\widehat{\mathrm{GT}}$ and GRT

- Alekseev-Torossion; Alekseev-Enriques-Torossion : There exists a pair of prounipotent groups KV and KRV

# Universal Finite Type Invariants or "Expansions":

$$\mathcal{K} \xrightarrow{\quad Z \quad} \mathrm{A}$$

- Bar-Natan : When $\mathcal{K} = \mathrm{PaB}$ and $\mathrm{A} = \mathrm{PaCD}$:

$$\{Z\} \rightleftarrows \{\text{Associators}\}.$$

- Drinfeld : There exists a pair of prounipotent groups $\widehat{\mathrm{GT}}$ and GRT

$$\widehat{\mathrm{GT}} \circlearrowright \mathrm{PaB} \xrightarrow{\quad Z \quad} \mathrm{PaCD} \circlearrowleft \mathrm{GRT}$$

- Bar-Natan, Fresse,... : There exist isomorphisms

$$\widehat{\mathrm{GT}} \cong \mathrm{Aut}_0(\widehat{\mathrm{PaB}}) \quad \text{and} \quad \mathrm{GRT} \cong \mathrm{Aut}_0(\mathrm{PaCD}).$$

# Expansions and The Kashiwara-Vergne Conjecture:

- Homomorphic Expansions of *w*-**foams**

$$\{\mathrm{wF} \xrightarrow{Z} \mathrm{A}\} \longleftrightarrow \{SolKV\}$$

- There exists a pair of prounipotent groups KV and KRV:

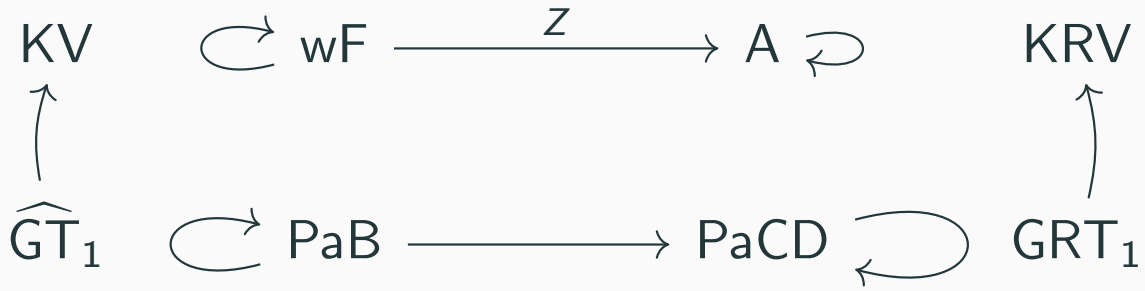
$$\mathrm{KV} \quad \circlearrowright \ \mathrm{wF} \xrightarrow{Z} \mathrm{A} \ \circlearrowright \quad \mathrm{KRV}$$

- Dancso-Halacheva-R : There exist isomorphisms

$$\mathrm{KV} \cong \mathrm{Aut}_v(\widehat{\mathrm{wF}}) \quad \text{and} \quad \mathrm{KRV} \cong \mathrm{Aut}_v(\mathrm{A}).$$

# The Kashiwara-Vergne Conjecture:

- Solutions to the KV conjecture are closely related to quantised deformation theory.




- Dancso-Halacheva-R: There exists an isomorphism

$$GRT_1 \cong \mathrm{Aut}_v^b(A).$$

# Circuit Algebras:

**Definition:** A **circuit algebra** is an algebra over the operad of **wiring diagrams**.

**Theorem (Dancso-Halacheva-R)**
*There is an equivalence of categories:*

$$\{\text{oriented circuit algebras}\} \cong \{\text{wheeled props}\}$$

# Wiring Diagrams:

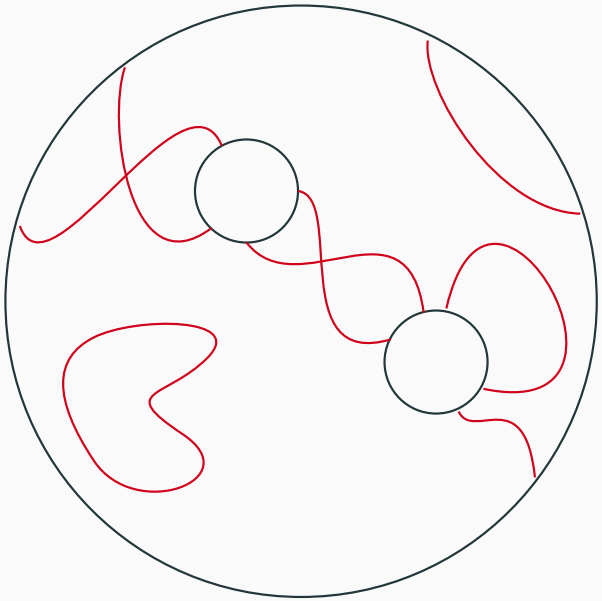
A **wiring diagram** consists of:

(1) A disc with $n$-holes: $D_0 \setminus \{\mathring{D}_1 \cup \dots \mathring{D}_n\}$

(2) A set of distinguished (input and output) labels $L_k$ on $\partial D_k$, $0 \leq k \leq n$.

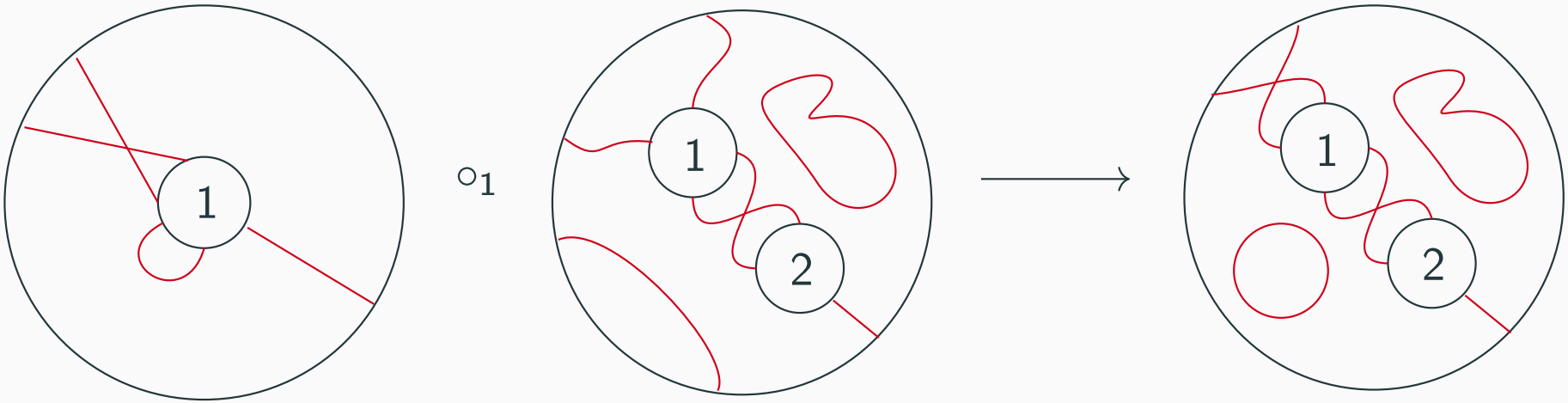
(3) An immersed (oriented) 1-manifold $M$ together with a bijection

$$\partial M \rightleftarrows \bigcup_k \partial D_k$$



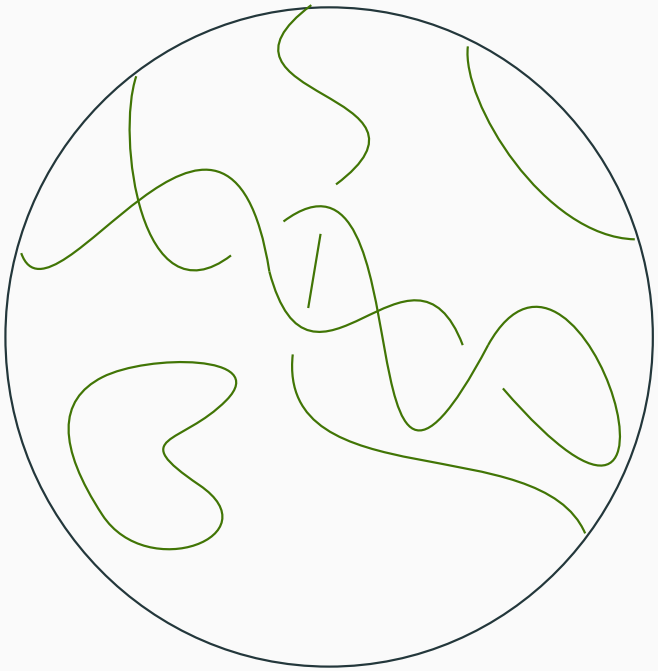
# Wiring Diagrams:

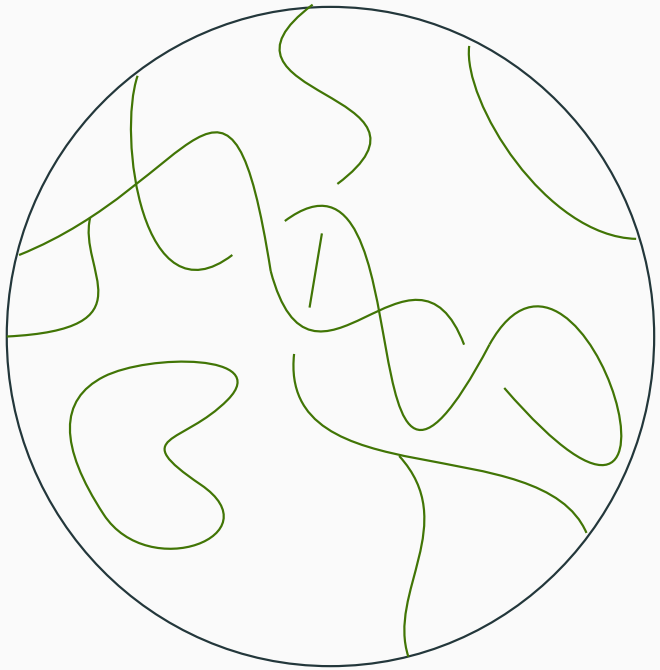
The collection of all wiring diagrams forms a **coloured operad** WD



# The circuit algebra of $w$-foams

- $w$-**tangles** are embeddings of (framed) surfaces into $\mathbb{R}^4$ up to ambient isotopy.
- $w$-**foams** are $w$-tangles where we allow "foamed vertices" – intersections of two tubes which may "merge" or "split."



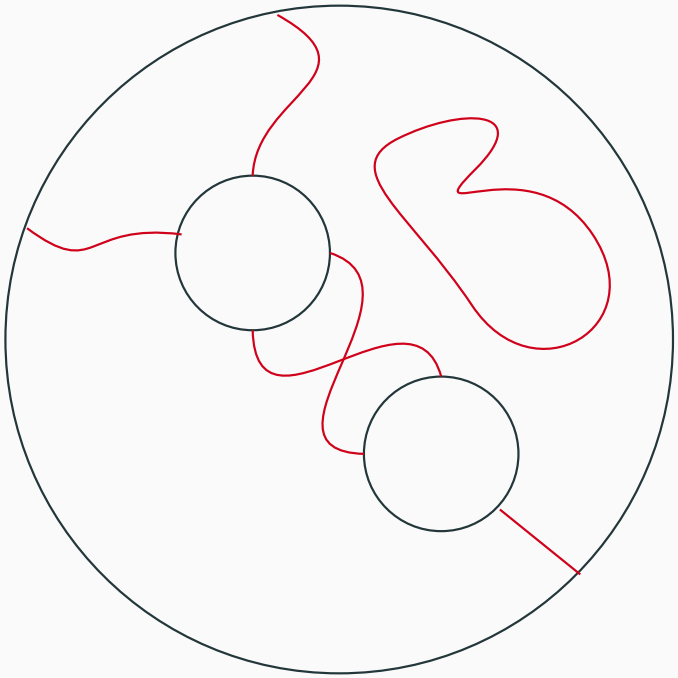


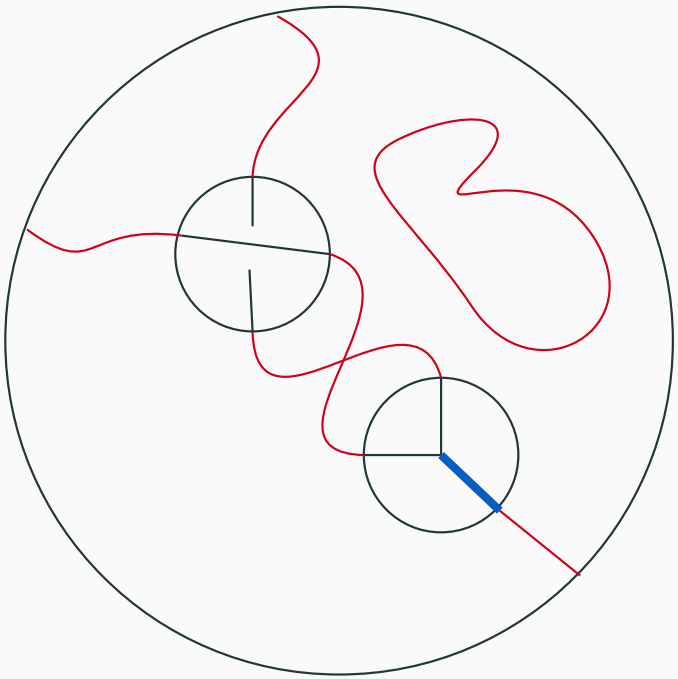
# The circuit algebra of $w$-foams

The circuit algebra of $w$- **foams** is:

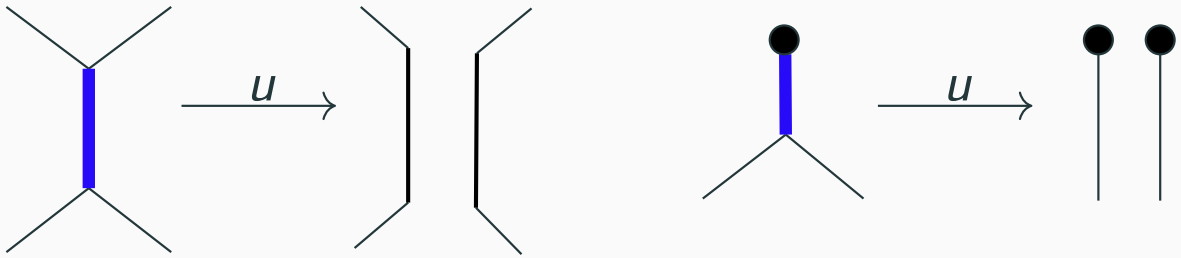
$$\mathrm{wF} = \mathrm{CA}\left\langle \; , \; , \; , \; \mid R1^s, R2, R3, R4, OC, CP \right\rangle .$$

Moreover, wF is equipped with auxiliary operations: $S_e, A_e, d_e$ and **unzips** $u_e$.





# The circuit algebra of $w$-foams

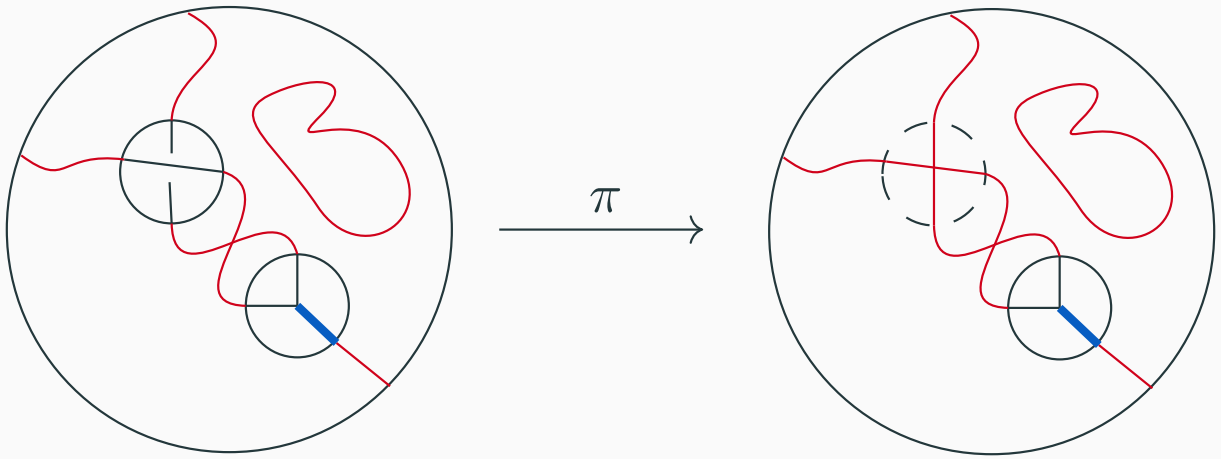


# The $w$-foam skeleton

The circuit algebra of $w$-foam **skeleta** is the circuit algebra

$$\mathcal{S} = \mathrm{CA}\left\langle \; , \; \right\rangle$$

with auxiliary operations: $S_e, A_e, u_e$ and $d_e$.




$$\mathrm{wF} := \coprod_{s \in \mathcal{S}} \pi^{-1}(s) = \coprod_{s \in \mathcal{S}} \mathrm{wF}(s).$$

# Completion of $w$-foams

Let $\mathbb{Q}[\mathrm{wF}](s)$ denote the $\mathbb{Q}$-vector space of formal linear combinations of $w$-foams $T_i$ with skeleton $s \in \mathcal{S}$,

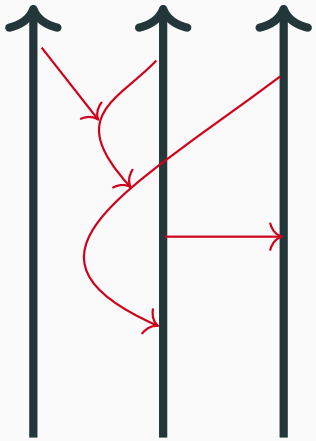
$$\sum_{T_i \in \mathrm{wF}(s)} \alpha_i T_i \in \mathbb{Q}[\mathrm{wF}](s).$$

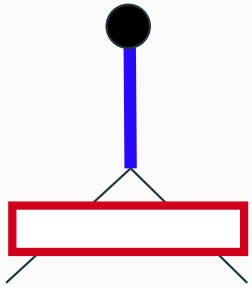
The (prounipotent) **completion** $\widehat{\mathrm{wF}} = \coprod_{s \in \mathcal{S}} \widehat{\mathrm{wF}}(s)$ where

$$\widehat{\mathrm{wF}}(s) := \lim \left( \mathbb{Q}[\mathrm{wF}](s)/\mathcal{I}(s) \leftarrow \mathbb{Q}[\mathrm{wF}](s)/\mathcal{I}^2(s) \leftarrow \mathbb{Q}[\mathrm{wF}](s)/\mathcal{I}^3(s) \leftarrow \ldots \right)$$

# The associated graded: arrow diagrams

**Arrow diagrams** are oriented Jacobi diagrams on $w$-foam skeleton.



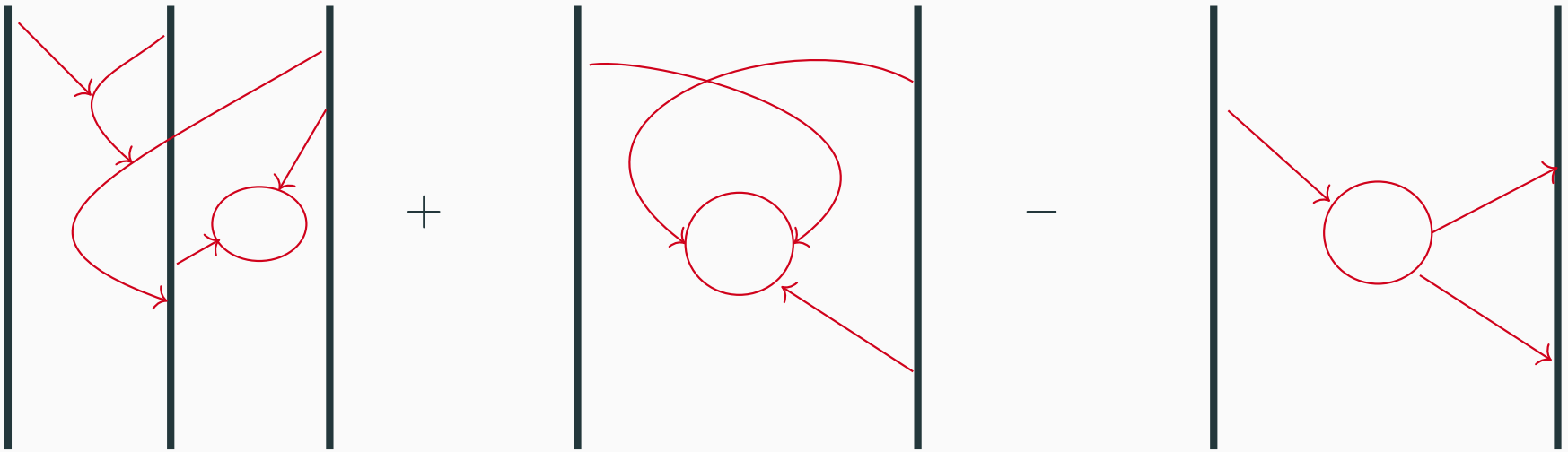


# The associated graded: arrow diagrams

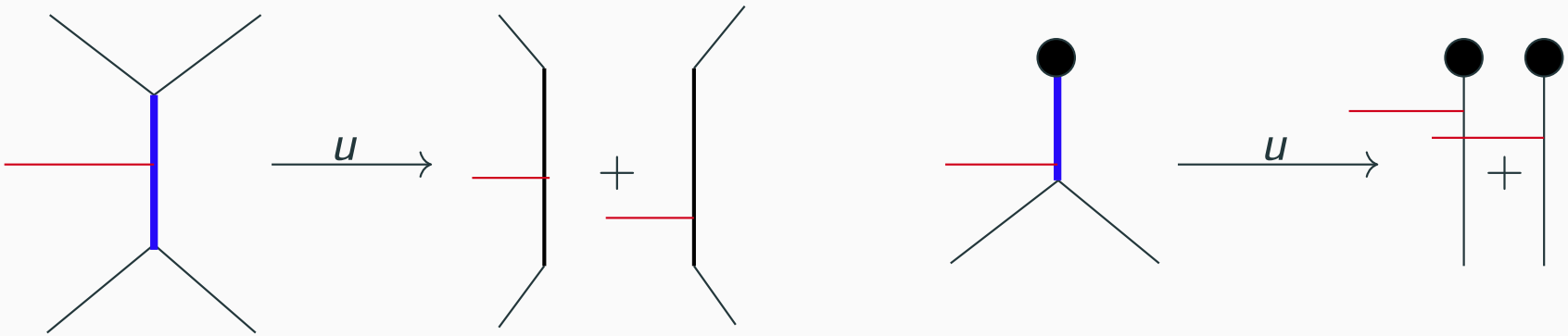
The circuit algebra of **arrow diagrams** is the complete circuit algebra in $\mathbb{Q}$-vector spaces

$$\mathsf{A} := \mathsf{CA}\left\langle \mapsto, \bullet, \curlywedge \mid \mathsf{4T}, \mathsf{TC}, \mathsf{VI}, \mathsf{CP}, \mathsf{RI} \right\rangle$$

with the associated graded of the $w$-foam skeleton operations: $S_e, A_e, d_e$, unzips $u_e$.



# The associated graded: arrow diagrams



# How does this relate to the KV solutions?

A **tangential derivation** on $\widehat{\mathfrak{lie}}_n$ is a derivation $u$ of $\widehat{\mathfrak{lie}}_n$ which acts on the generators $x_i$ by $u(x_i) = [x_i, a_i]$, for some $a_i \in \widehat{\mathfrak{lie}}_n$.

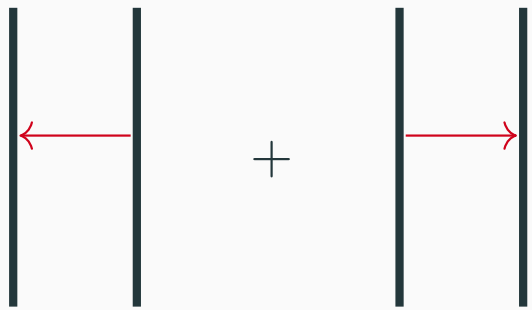
**Precisely:**

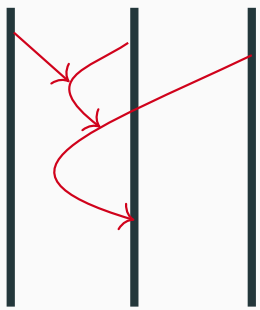
$$\mathsf{SolKV} = \mathsf{SolKV}(\mathbb{Q}) := \Big\{ (F, r) \in \mathsf{TAut}_2(\mathbb{Q}) \times u^2\mathbb{Q}[[u]] \Big| \\ F(e^x e^y) = e^{x+y} \text{ and } J(F) = \mathrm{tr}\Big( r(x+y) - r(x) - r(y) \Big) \Big\}.$$

# How does this relate to the KV solutions?

Ex (1) $t^{1,2} = (y, x) \in \mathfrak{tder}_2$.

Ex (2) $u = (0, [[x, y], z], 0) \in \mathfrak{tder}_3$

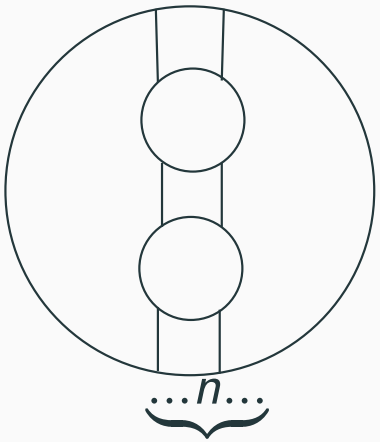




# How does this relate to the KV solutions?

There is an isomorphism of Hopf algebras

$$\mathsf{A}(\uparrow_n) \cong \hat{U}((\mathfrak{tder}_n \oplus \mathfrak{a}_n) \ltimes \mathrm{cyc}_n).$$



# Expansions

A **homomorphic expansion** of *w*-foams is a circuit algebra homomorphism $Z : \mathsf{wF} \to \mathsf{A}$ such that:

- its linear extension $Z : \mathbb{Q}[\mathsf{wF}] \to \mathsf{A}$ is a filtered, skeleton preserving homomorphism
- $\mathrm{gr}Z = \mathrm{id}_{\mathsf{A}}$.

**Theorem (Dancso-Halacheva-R)**
*For any homomorphic expansion the induced map* $\widehat{Z} : \widehat{\mathsf{wF}} \to \mathsf{A}$ *is an isomorphism.*

**Theorem (Bar-Natan – Dancso)**
*There is a one-to-one correspondence:*

$$\{\text{v-small exp}\} \rightleftarrows \{\text{SolKV}\}.$$

# The graded group KRV

$$\mathsf{KRV} = \mathsf{KRV}(\mathbb{Q}) := \Big\{(\alpha, s) \in \mathsf{TAut}_2(\mathbb{Q}) \times u^2\mathbb{Q}[[u]] \Big| \\ \alpha(e^{x+y}) = e^{x+y} \text{ and } J(\alpha) = \mathrm{tr}\Big(s(x+y) - s(x) - s(y)\Big)\Big\}$$

SolKV $\circlearrowright$ KRV is given by $F \cdot \alpha = \alpha^{-1} \circ F$.

**Theorem (Dancso-Halacheva-R)**
*There is an isomorphism of groups* $\mathsf{Aut}_v(\mathsf{A}) \cong \mathsf{KRV}$.

# The group KV

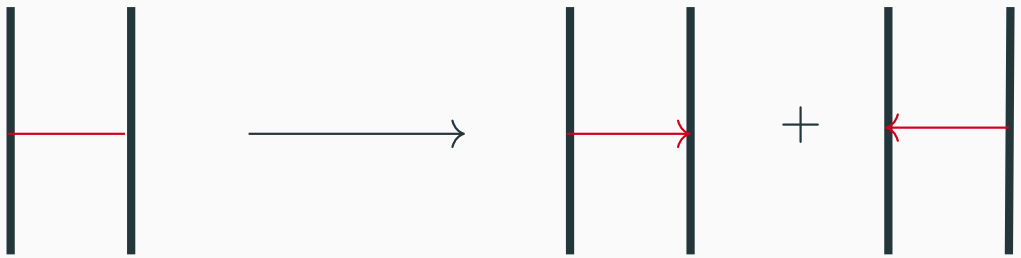
$$\mathsf{KV} = \mathsf{KV}(\mathbb{Q}) := \Big\{ (a,\sigma) \in \mathsf{TAut}_2(\mathbb{Q}) \times u^2\mathbb{Q}[[u]] \Big| \; a(e^x e^y) = e^x e^y \text{ and } J(a) = \mathrm{tr}\Big(\sigma(\mathfrak{bch}(x,y)) - \sigma(x) - \sigma(y)\Big) \Big\}$$

$\mathsf{KV} \circlearrowright \mathsf{SolKV}$ is given by $a \cdot F = F \circ a^{-1}$.

**Theorem (Dancso-Halacheva-R)**
*There is an isomorphism of groups* $\mathrm{Aut}_v(\widehat{\mathsf{wF}}) \cong \mathsf{KV}$.

# The relationship between KRV and $\mathrm{GRT}_1$

- **chord diagrams** are pictorial depictions of $\hat{U}(\mathfrak{t}_n)$
- There's an inclusion of Lie algebras $\mathfrak{t}_n \hookrightarrow \mathfrak{tder}_n$.

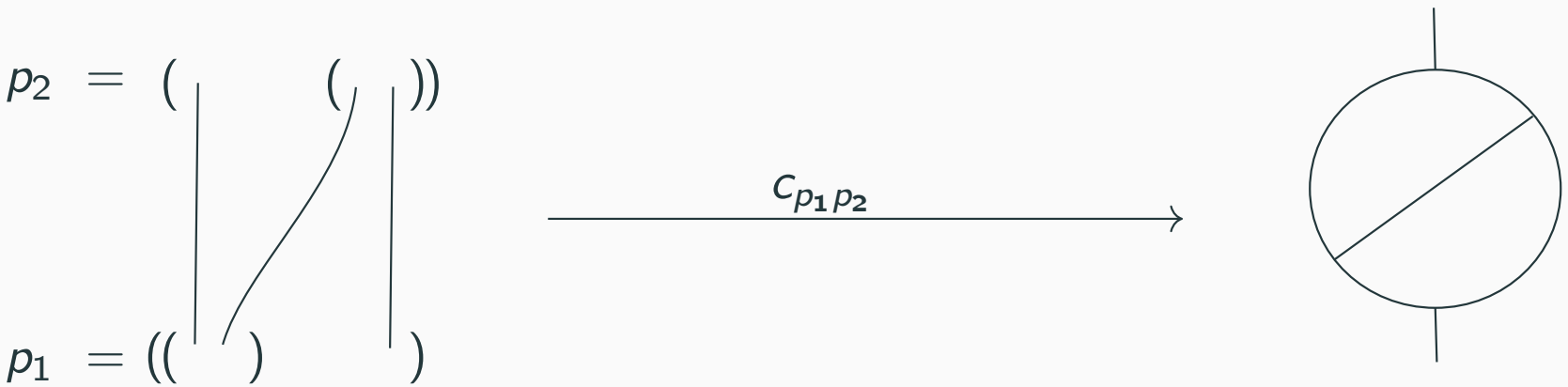


**Lemma (Dancso-Halacheva-R)**
*For each $n \geq 2$ there exists an inclusion of Hopf algebras*

$$\mathrm{Hom}_{\mathrm{CD}(n)}(*,*) \xrightarrow{\varepsilon} \mathrm{A}(\uparrow_n).$$

# The relationship between KRV and $\mathsf{GRT}_1$

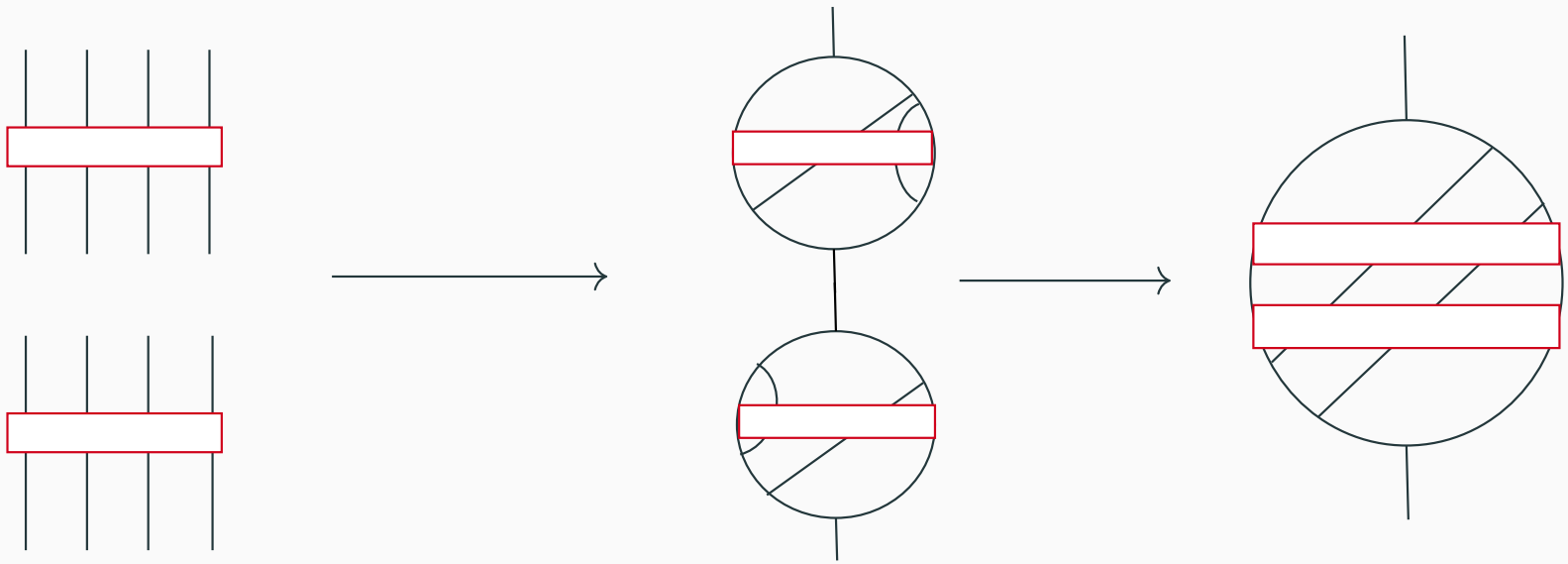
**Theorem (Bar-Natan, Fresse)**
*There is an isomorphism of groups* $\mathsf{Aut}_0(\mathsf{PaCD}) \cong \mathsf{GRT}_1$.



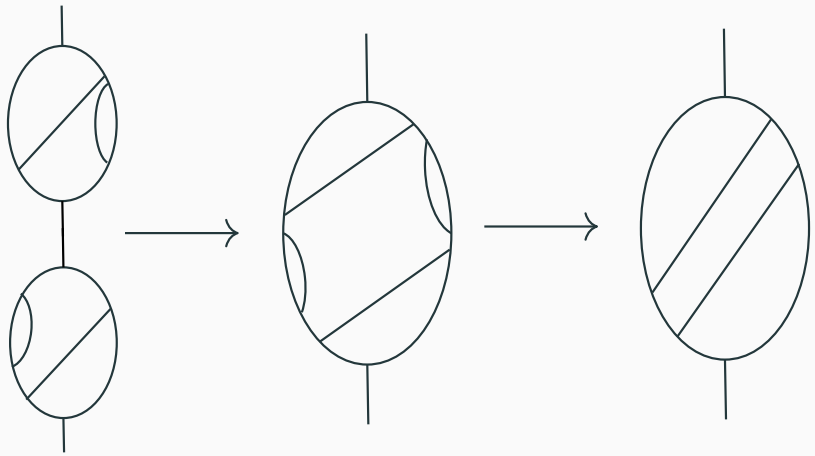
# The relationship between KRV and $\mathrm{GRT}_1$

**Theorem (Dancso-Halacheva-R)**
*There is an inclusion of Hopf groupoids*

$$\mathrm{PaCD}(n) \xrightarrow{\varepsilon} \bigcup_{c_{p_i p_j}} \mathrm{A}(c_{p_i p_j}).$$



# The pentagon



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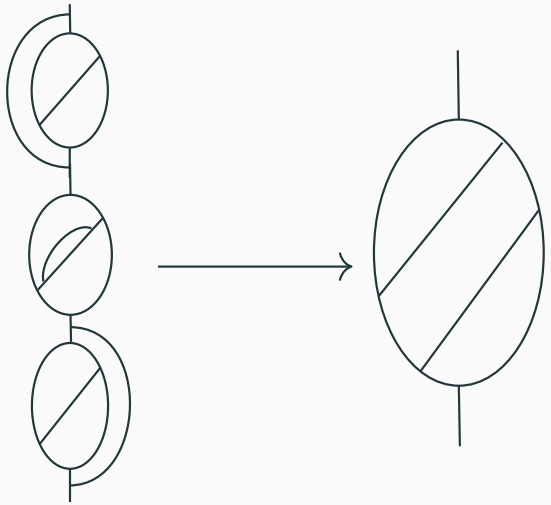

# The relationship between KRV and $\mathrm{GRT}_1$

**Theorem (Dancso-Halacheva-R)**

*The (isomorphic) image of* $\mathrm{GRT}_1$ *in* $\mathrm{Aut}_v(\mathrm{A})$ *is the subgroup*

$$\mathrm{E} = \{\tilde{G} \in \mathrm{Aut}_v(\mathrm{A}) \ : \ \tilde{G}(\oslash) = \varepsilon(G(\mathbin{|\!\!\diagup\!\!|})) \text{ for some (unique) } G \in \mathrm{Aut}_0(\mathrm{PaCD})\}.$$

# Thanks!