Automorphisms of seemed surfaces, modular operads and Galois actions

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A really fast introduction to a lot of cool math:

- Let $Gal(\mathbb{Q})$ denote the absolute Galois group of $\mathbb{Q}$.
- This is a large profinite group:

$$\widehat{G} = \lim G/H$$

  but we don't even know the finite quotients of $Gal(\mathbb{Q})$!

Idea : Identify $g \in Gal(\mathbb{Q})$ with a pair

$$(\chi(g), f_g) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$$

  - $\chi(g)$ is the **cyclotomic character**.
  - $\widehat{F}_2 = \pi_1(\mathcal{M}_{0,4}) \cong \widehat{\Gamma}_{0,4}$.

A slightly easier group: $\widehat{\mathrm{GT}}$

Notation: For any homomorphism of profinite groups

$$\widehat{F}_2 \longrightarrow G$$

$$(x, y) \longmapsto (a, b)$$

we write $f(a, b)$ for the image of any $f \in \widehat{F}_2$. For example:

- Given $id : \widehat{F}_2 \to \widehat{F}_2$, we have $f = f(x, y)$;
- Given the map $\widehat{F}_2 \to \widehat{F}_2$ which swaps generators x and y we have $f \mapsto f(y, x)$.

A slightly easier group: $\widehat{\mathrm{GT}}$

The **Grothendieck-Teichmüller group** $\widehat{\mathrm{GT}}$ is the group of pairs

$$(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$$

satsfying the property that

$$x \mapsto x^\lambda \quad \text{and} \quad y \mapsto f^{-1} y^\lambda f$$

induce an automorphism of $\widehat{F}_2$ and :

(I) $f(x,y)f(y,x) = 1$,

(II) $f(x,y)x^m f(z,x)z^m f(y,z)y^m = 1$ where $xyz = 1$ and $m = (\lambda - 1)/2$,

(III) $f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51})f(x_{12}, x_{23}) = 1$ in $\widehat{\Gamma}_{0,5}$ where x_{ij} is a Dehn twist of boundaries i and j.

Theorem (Ihara)
There is an injection $Gal(\mathbb{Q}) \hookrightarrow \widehat{\mathrm{GT}}$.

So our question becomes

- What is $\widehat{\mathrm{GT}}$?

- ** Can we refine $\widehat{\mathrm{GT}}$ and thus get closer to $Gal(\mathbb{Q})$?

Goals:

Construct a **modular operad** S in groupoids with the following properties:

(1) Each $\mathsf{S}(g, n)$ should "approximate" the mapping class group $\Gamma_{g,n}$, i.e. $B\mathsf{S}(g, n) \simeq B\Gamma_{g,n}$

(2) The group $Gal(\mathbb{Q}) \subseteq \Lambda \subseteq \widehat{\mathsf{GT}}$ defined by Nakamura-Schneps acts on the (profinite completion of the) modular operad S and

$$\mathrm{End}_0(\widehat{\mathsf{S}}) \cong \Lambda.$$

(3*) The action of the $Gal(\mathbb{Q})$ on $\widehat{B\mathsf{S}} \simeq \bar{\mathcal{M}}_{*,*}$ is non-trivial.

Modular Operads:

A **modular operad** P in a base symmetric monoidal category $\mathcal{E}$ is:

(1) A bi-graded Σ-module $\mathsf{P} = \{\mathsf{P}(g,n)\}$ in $\mathcal{E}$

(2) associative, equivariant **compositions**

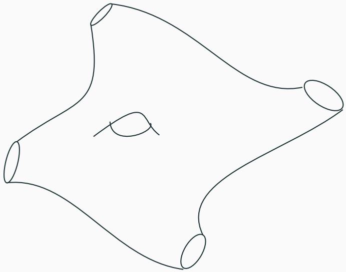
$$\mathsf{P}(g,n) \times \mathsf{P}(h,m) \xrightarrow{\circ_{ij}} \mathsf{P}(g+h, n+m-2)$$

(3) associative, equivariant **contractions**

$$\mathsf{P}(g,n) \xrightarrow{\xi_{ij}} \mathsf{P}(g+1, n-2)$$

Surfaces $\Sigma_{g,n}$.

Let $\Sigma_{g,n}$ denote a Riemann surface with *n*-boundaries:



The **mapping class group** of a surface Σ:

$$\Gamma_{g,n} = \pi_0 \mathrm{Diff}^+(\Sigma, \partial).$$

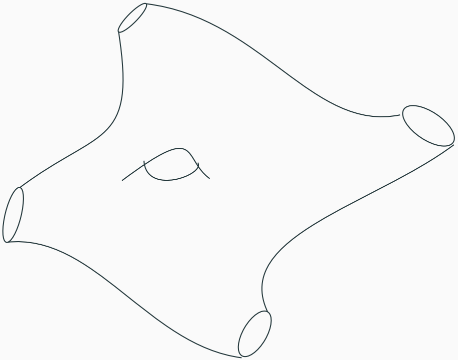
If you want to get ahead:

$$\widehat{\Gamma}_{g,n} \cong \pi_1(\mathcal{M}_{g,n}).$$

A nice fact about $\Gamma_{g,n}$

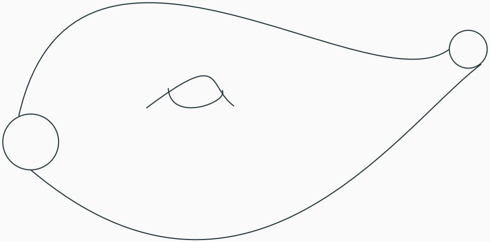
Hatcher-Thurston: The mapping class group has a presentation

$$\Gamma_{g,n} = \langle b_1, \ldots, b_n, a_1, .., a_k \mid (C), (B), (D), (L) \rangle \, .$$



Pants Decompositions

A **pants decomposition** of $\Sigma_{g,n}$ is a collection of simple closed curves that cuts $\Sigma_{g,n}$ into *pairs of pants* (i.e. $\Sigma_{0,3}$).

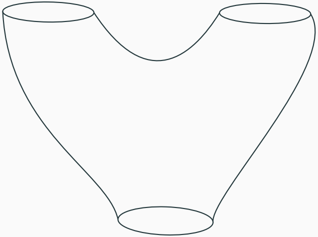


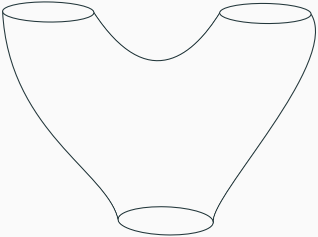
Quilted Pants Decompositions

We actually want **quilted** pants decompositions.

A **quilt** consists of:

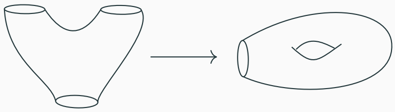
- 2 marked points, called *vertices*, for every circle in the decomposition;
- 3 disjoint lines between the vertices for each pair of pants;





Quilted Pants Decompositions

A quilted **pants decomposition** can be determined by modular operadic composition.



A groupoid of quilted surfaces

The goupoid $\mathsf{S}(g,n)$:

- objects are pairs (P,Q) giving an "atomic" quilt decomposition of $\Sigma_{g,n}$;
- morphisms are $\pi_0\mathrm{Diff}^+(\Sigma_{g,n},\partial,\sigma)$.

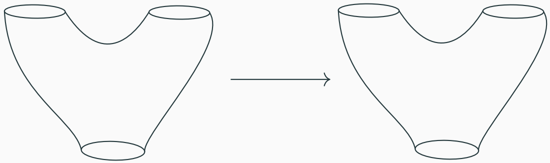
$\Rightarrow$

$$B\mathsf{S}(g,n)\simeq B\Gamma_{g,n}.$$

A groupoid of quilted surfaces

The morphisms in $\mathrm{S}(g, n)$ are a composite of three types of elementary morphisms:

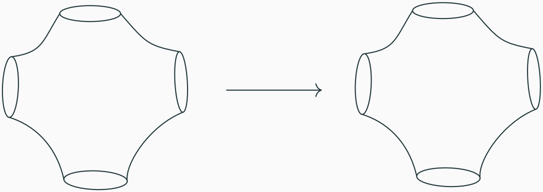
- $\frac{1}{2}$ – Dehn twists



A groupoid of quilted surfaces

The morphisms in $\mathsf{S}(g, n)$ are a composite of three types of elementary morphisms:

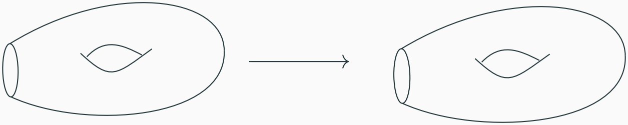
- *A*-moves



A groupoid of quilted surfaces

The morphisms in $\mathsf{S}(g,n)$ are a composite of three types of elementary morphisms:

- *S*-moves



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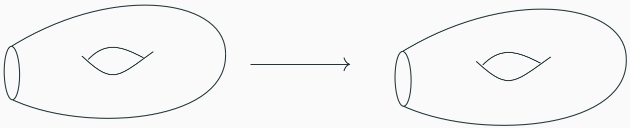


A modular operad of Seamed Surfaces

The elementary morphisms in $\mathsf{S}(g, n)$ satisfy 6 important relations:

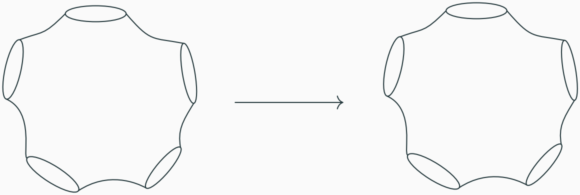
$$A_{\beta_{\mathbf{3}},\beta_{\mathbf{4}}} A_{\beta_{\mathbf{1}},\beta_{\mathbf{2}}} A_{\beta_{\mathbf{4}},\beta_{\mathbf{5}}} A_{\beta_{\mathbf{2}},\beta_{\mathbf{3}}} A_{\beta_{\mathbf{5}},\beta_{\mathbf{1}}} = 1 \tag{5A}$$

$$S_{\beta_{\mathbf{1}},\beta_{\mathbf{2}}} S_{\beta_{\mathbf{2}},\beta_{\mathbf{3}}} S_{\beta_{\mathbf{3}},\beta_{\mathbf{1}}} = 1 \tag{3S}$$

$$A_{\alpha_{\mathbf{3}},\epsilon_{\mathbf{3}}} S_{\alpha_{\mathbf{1}},\alpha_{\mathbf{2}}} A_{\epsilon_{\mathbf{3}},\epsilon_{\mathbf{2}}} A_{\epsilon_{\mathbf{2}},\epsilon_{\mathbf{1}}} S_{\alpha_{\mathbf{2}},\alpha_{\mathbf{3}}} A_{\epsilon_{\mathbf{1}},\alpha_{\mathbf{1}}} = 1. \tag{6AS}$$

Relations on $\mathsf{S}(g, n)$

The A-move A_{β_3,β_4} is a diffeomorphism from the pants decomposition containing β_3 to the decomposition containing β_4:



A modular operad of Seamed Surfaces

Operations:

$$\mathsf{S}_{g,n} \times_{ij} \mathsf{S}_{h,k} \xrightarrow{\circ_{ij}} \mathsf{S}_{g+h,n+k-2}$$

and

$$\mathsf{S}_{g,n} \xrightarrow{\xi_{ij}} \mathsf{S}_{g+1,n-2}$$

can be defined on objects by gluing surfaces and on morphisms as the "combination" of the maps on the subsurfaces.

These are well-defined, associative operations and thus

$$\mathsf{S} = \{\mathsf{S}_{g,n}\}$$

assembles into a **modular operad in groupoids**.

The connection to $\widehat{\mathsf{GT}}$

Profinite $\widehat{\mathrm{GT}}$

The **Grothendieck-Teichmüller group** $\widehat{\mathrm{GT}}$ is the group of pairs

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satsfying the property that

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induce an automorphism of $\widehat{F}_2$ and :

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Theorem (Ihara)
There is an injection $Gal(\mathbb{Q}) \hookrightarrow \widehat{\mathrm{GT}}$.

The genus 0 case

Proposition (Bovida-Horel-R)
There's an action of $\widehat{\mathrm{GT}}$ on the genus 0 part of our operad, $\widehat{\mathrm{S}_0}$.

Proposition (Bovida-Horel-R)

$$\widehat{\mathrm{GT}} \cong \mathrm{End}_0(\widehat{\mathrm{S}_0})$$

This uses in a fundamental way:

Proposition (Horel)
Let C and D be two groupoids (with finite sets of objects). The map

$$\widehat{C \times D} \to \widehat{C} \times \widehat{D}$$

is an isomorphism.

$\Rightarrow$ The profinite completion $\widehat{\mathrm{S}_0} = \{\widehat{\mathrm{S}(0,n)}\}$ is an operad in profinite groupoids.

Groups Related to $\widehat{\mathrm{GT}}$: The higher genus case

Definition (Nakamura-Schneps)
Let Λ denote the group of pairs

$$(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$$

which satisfy the property that

$$x \mapsto x^\lambda \quad \text{and} \quad y \mapsto f^{-1} y^\lambda f$$

induce an automorphism of $\widehat{F}_2$ and relations $(I) - (III)$ of $\widehat{\mathrm{GT}}$ and :

(IV)

$$f(e_1, a_1) a_3^{-8\rho_2} f(a_2^2, a_3^2)(a_3 a_2 a_3)^{2m} f(e_2, e_1) e_2^{2m} f(e_3, e_2) \\ a_2^{-2m}(a_1 a_2 a_1)^{2m} f(a_1^2, a_2^2) a_1^{8\rho_2} f(a_3, e_3) = 1$$

where a_1, a_2, a_3, e_1, e_2 are Dehn twists in $\Gamma_{1,2}$, $m = (\lambda - 1)/2$ and ρ_2 is an integer.

Groups Related to $\widehat{GT}$: The higher genus case

(IV)

$$f(e_1,a_1)a_3^{-8\rho_2}f(a_2^2,a_3^2)(a_3a_2a_3)^{2m}f(e_2,e_1)e_2^{2m}f(e_3,e_2)$$
$$a_2^{-2m}(a_1a_2a_1)^{2m}f(a_1^2,a_2^2)a_1^{8\rho_2}f(a_3,e_3) = \quad 1$$

where a_1, a_2, a_3, e_1, e_2 are Dehn twists in $\Gamma_{1,2}$, $m = (\lambda-1)/2$ and ρ_2 is an integer.

Nakamura-Schneps: $Gal(\mathbb{Q}) \subseteq \Lambda \subseteq \widehat{GT}$.

Horel's result applies to the modular operad $\mathsf{S} = \{\mathsf{S}(g, n)\}$ and

$$\widehat{\mathsf{S}} = \{\widehat{\mathsf{S}(g, n)}\}$$

is a modular operad in profinite groupoids.

Proposition (Bonatto-R)
There is an action

$$\Lambda \longrightarrow \mathrm{End}_0(\widehat{\mathsf{S}}).$$

Given a pair $(\lambda, f) \in \Lambda$:

$$(\lambda, f) \longrightarrow \begin{cases} a_{\frac{1}{2}} \mapsto a_{\frac{1}{2}}^{\lambda} \\ A_{\alpha,\beta} \mapsto A_{\alpha,\beta} \cdot f(a, b) a^{N(\lambda-1)/2} \\ S_{\alpha,\beta} \mapsto S_{\alpha,\beta} \cdot (aba)^{\lambda-1} b^{N(\lambda-1)/2-8\rho_2} f(a^2, b^2) a^{8\rho_2} \end{cases}$$

where N is some integer (a "twist") and ρ_2 is a different integer.

Let's check this is well-defined

Recall that the S-move $S_{\alpha,\beta}$ is the diffeomorphism:

Let's check this is well-defined

We just need to check that the action

$$S_{\alpha,\beta} \mapsto S_{\alpha,\beta} \cdot (aba)^{\lambda-1} b^{N(\lambda-1)/2-8\rho_{\mathbf{2}}} f(a^2, b^2) a^{8\rho_{\mathbf{2}}}$$

interacts well with the relations on the morphisms of the groupoids $\mathcal{S}_{g,n}$, eg:

$$A_{\alpha_{\mathbf{3}},\epsilon_{\mathbf{3}}} S_{\alpha_{\mathbf{1}},\alpha_{\mathbf{2}}} A_{\epsilon_{\mathbf{3}},\epsilon_{\mathbf{2}}} A_{\epsilon_{\mathbf{2}},\epsilon_{\mathbf{1}}} S_{\alpha_{\mathbf{2}},\alpha_{\mathbf{3}}} A_{\epsilon_{\mathbf{1}},\alpha_{\mathbf{1}}} = 1 \qquad \text{(6AS)}$$

Let's check this is well-defined

$$A_{\alpha_3,\epsilon_3} S_{\alpha_1,\alpha_2} \dots A_{\epsilon_1,\alpha_1} \mapsto$$

$$A_{\alpha_3,\epsilon_3} f(a_3, e_3) a_3^{N(\lambda-1)/2} S_{\alpha_1,\alpha_2} (a_1 a_2 a_1)^{\lambda-1} a_2^{N(\lambda-1)/2-8\rho_2} f(a_1^2, a_2^2) a_1^{8\rho_2} \dots$$
$$\dots A_{\epsilon_1,\alpha_1} f(e_1, a_1) e_1^{N(\lambda-1)/2}$$

The integers N are computed based on the quilting.

This yields the relation (IV):

$$f(e_1, a_1) a_3^{-8\rho_2} f(a_2^2, a_3^2)(a_3 a_2 a_3)^{2m} f(e_2, e_1) e_2^{2m} \dots$$
$$f(e_3, e_2) a_2^{-2m} (a_1 a_2 a_1)^{2m} f(a_1^2, a_2^2) a_1^{8\rho_2} f(a_3, e_3) = 1$$

Is this action interesting?

The spaces $B\mathcal{S}(0,n)$ are very closely related to the moduli space $\bar{\mathcal{M}}_{0,n}$:

Theorem (Drummond-Cole)
There's a homotopy pushout of operads :

$$\begin{array}{ccc} S^1 & \longrightarrow & B\mathcal{S}(0,n) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \bar{\mathcal{M}}_{0,n} \end{array}$$

The idea:

- Use $\widehat{\mathrm{GT}} \cong \mathrm{End}_0(\widehat{\mathrm{S}_0}) = \pi_0(\mathbb{R}\,\mathrm{End}(\widehat{B\mathcal{S}_0}))$ to get an action of $\widehat{\mathrm{GT}}$ (and thus $Gal(\mathbb{Q})$) on $\widehat{B\mathcal{S}(0,n)}$
- In this case, profinite completion interacts well with pushouts of ∞-operads $\Rightarrow$ the Galois action is non-trivial.

Groups Related to $\widehat{\mathsf{GT}}$: What do we know in the higher genus case ?

Recall:

(1) $Gal(\mathbb{Q}) \subseteq \Lambda \subseteq \widehat{\mathsf{GT}}$

(2) The spaces $B\mathsf{S}(g, n) \simeq B\Gamma_{g,n}$ are closely related to the spaces $\bar{\mathcal{M}}_{g,n}$

To study the action of $Gal(\mathbb{Q})$ on the modular operad $\widehat{B\mathsf{S}}$:

- We no longer have the ability to compare with ribbon braids in genus 0;
- The classifying spaces $B\mathcal{S}_{g,n}$ are not "nice" for $g \geq 2$ and so the best you can do:

Theorem (Bonatto-R)

$$\pi_0 \mathbb{R}End(\widehat{NB\mathcal{S}}) \simeq \pi_0 \mathbb{R}End(\widehat{NB\mathcal{S}}_{\leq 1})$$