Profinite Completion of Modular Operads and an ∞-Cyclic Operad of Surfaces

Ethan Armitage

Supervisor: Marcy Robertson

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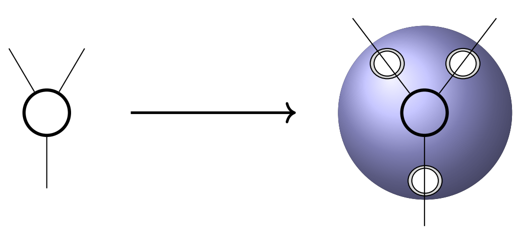

School of Mathematics and Statistics
Unviersity of Melbourne
Australia
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Chapter 1

Introduction

1.1 Grothendieck's *esquisse d'un programme*

In 1984 Grothendieck submitted his research proposal '*esquisse d'un programme*', [Gro97], for a position at the Centre national de la recherche scientifique. It outlined a program to study the absolute galois group of the rationals, $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, via its action on geometric spaces.

The idea comes from the fact that given a scheme X defined over $\mathbb{Q}$, there is a natural action of $G_{\mathbb{Q}}$ on the scheme of geometric points $X \times_{\mathbb{Q}} \overline{\mathbb{Q}}$. Grothendieck singled out the moduli space of genus g Riemann surfaces with n boundary components, $\mathcal{M}_{g,n}$, as being of particular interest. He also decided to not study the action of $G_{\mathbb{Q}}$ on the space itself, but rather on its étale fundamental group $\hat{\pi}_1(\mathcal{M}_{g,n} \times_{\mathbb{Q}} \overline{\mathbb{Q}})$.

Recall that the étale fundamental group is defined as a limit of automorphism groups of covering maps. Hence moving to the fundamental group allowed Grothendieck to change the problem from being about spaces to a problem about covering maps. This allowed for the application of technology such as Belyi's theorem:

Theorem 1.1.1. *A non-singular complex algebraic curve X is defined over $\mathbb{Q}$*

if, and only if, there is a branched covering $X \to \mathbb{C}P^1$ *ramified at* $\{0,1,\infty\}$. *In other words, if, and only if, there is a covering space map* $X \to \mathbb{C}P^1 \backslash \{0,1,\infty\}$.

See [Bel80] for a proof. Recalling that the moduli space $\mathcal{M}_{0,4}$ is in fact given by $\mathbb{C}P^1$, Belyi's theorem gives us a door to access the action of $G_{\mathbb{Q}}$ on $\hat{\pi}_1(\mathcal{M}_{0,4} \times_{\mathbb{Q}} \overline{\mathbb{Q}})$. Specifically, it allows us to prove that this action is faithful. This can be seen by considering the corresponding action on elliptic curves and their j-invariant.

Thus we have established that $G_{\mathbb{Q}}$ is a subgroup of $Aut(\hat{\pi}_1(\mathcal{M}_{0,4} \times_{\mathbb{Q}} \overline{\mathbb{Q}})) \cong Aut(\hat{F}_2)$, where $\hat{F}_2$ is the profinite completion of the free group on two letters (see Definition 4.2.2 for the definition of profinite completion). It is too much to expect this inclusion to actually be an isomorphism (and indeed this is not true).

However, Grothendieck then suggested we examine the action of $G_{\mathbb{Q}}$ on the collection of all étale fundamental groups $\hat{\pi}_1(\mathcal{M}_{g,n} \times_{\mathbb{Q}} \overline{\mathbb{Q}})$, along with certain 'nice' maps between them induced by inclusion of subsurfaces. This is known as the Teichmüller tower, which we denote by $\mathcal{T}$. Adding in this extra structure should shrink the automorphism group, getting us closer to $G_{\mathbb{Q}}$.

Note that the action of $G_{\mathbb{Q}}$ on this tower is faithful as it is faithful on the level $\hat{\pi}_1(\mathcal{M}_{0,4} \times_{\mathbb{Q}} \overline{\mathbb{Q}})$. Hence $G_{\mathbb{Q}}$ embeds in $Aut(\mathcal{T})$. Grothendieck conjectured that this is in fact an isomorphism.

One large problem is that we do not know what the exact definition of a 'nice' map should be. Morally, the correct definition should be whatever makes Grothendieck's conjecture true, but there are many proposed models for $\mathcal{T}$. One approach, which is the approach we adopt in this thesis, is to model $\mathcal{T}$ using the theory of operads. The key insight for this approach is that inclusion of a subsurface $F \hookrightarrow S$ can be viewed as giving a decomposition of S as the result of gluing the surfaces F and $\overline{F^c}$ along their boundary.

An operad can be viewed as a collection of objects in a category $\mathcal{C}$, parametrised by trees, along with morphisms between these objects. These morphisms should in a rigorous way model the operation of gluing trees together. Note that we can view a surface as being a tree with one vertex and the same number of legs as boundary components. Gluing together surfaces along boundary components is entirely analogous to gluing trees along legs, giving a link between operads and low dimensional topology.

This is, however, limited. For one, this approach completely forgets any genus data. For another, given two trees T_1 and T_2 the theory of operads only allows us to glue one leg of T_2 to one leg of T_1. For instance, in this formalism we cannot model the operation of gluing two boundary components of the same surface together, increasing the genus by 1. This is an important operation and should be included in the definition of the Teichmüller tower.

The limitations of operads lead us to the theory of modular operads. These are in a rigorous sense normal operads with an extra genus structure and contraction maps that increase genus. In Chapter 2 we define modular operads and review their basic theory.

Another problem that arises is that we need to understand how operads and their modular counterparts interact with the operation of profinite completion. It turns out that the profinite completion of a (modular) operad will not in general be a (modular) operad. However, in certain nice cases they will be (modular) operads 'up to homotopy', henceforth referred to as ∞(-modular) operads.

To model this concept, we redefine modular operads as presheafs on a certain graphical category defined in Chapter 3. Then in Chapter 4 we formally define our model of ∞-modular operads and describe some situations in which the profinite completion of a strict modular operad can give an ∞-modular operad.

Next, in chapter 5 we define the modular operad we want to use to construct a model of $\mathcal{T}$. However, we are not able to see if this operad fits into the 'nice' cases in which profinite completion gives an ∞-modular operad. This is due to a hard problem about the topology of surfaces. We will however show in Chapter 6 that these criteria hold in genus 0, which allows us to show that the profinite completion of an associated cyclic operad is ∞.

1.2 The nerve theorem for simplicial sets and ∞-categories

Recall the definition of the *simplex category* Δ: Its objects are given by sets of the form $[n] = \{0 < 1 < ... < n\}$ and its morphisms are given by order preserving functions. A *simplicial set* is then a functor $X : \Delta^{op} \to \mathbf{Set}$. It is a standard fact (see, for instance, [Wei94]) that the morphisms of Δ are generated by a special class of functions called *codegeneracies* and *coface maps*:

Definition 1.2.1. A *codegeneracy* $s^i : [n+1] \to [n]$ is given by

$$s^i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$$

Similarly a *coface map* $d^i : [n-1] \to [n]$ is given by

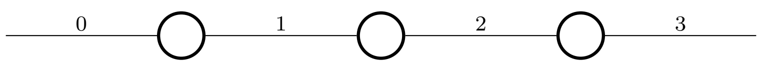
$$d^i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

These morphisms also satisfy the cosimplicial identities. Hence to define a simplicial set it suffices to give a collection of sets $\{X_n | n \in \mathbb{Z}_{\geq 0}\}$ with degeneracies $s_i : X_n \to X_{n+1}$ and face maps $d_i : X_n \to X_{n-1}$ satisfying the simplicial identities, which are obtained by dualizing the cosimplicial identities. Explicitly, the face and degeneracy maps must satisfy the following relations:

$$\begin{cases} d_i d_j = d_{j-1} d_i & i < j \\ d_i s_j = s_{j-1} d_i & i < j \\ d_j s_j = 1 = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & i > j+1 \\ s_i s_j = s_{j+1} s_i & i \leq j+1 \end{cases}$$

It is possible to view the objects of Δ as being graphs. We can represent $[n]$ as being a linear graph with n vertices, $n+1$ edges and no branching. See the example below:

Example 1.2.2.



This is a graph representing the object $[3] \in \Delta$.

This in fact gives a natural description of Δ^{op}. The degeneracy s_i is given by inserting a vertex between the ith and $(i+1)$th vertices. Similarly, the face map d_i is given by contracting the edge between the ith and $(i+1)$th vertices.

Simplicial sets are a natural setting for abstract homotopy theory. For us, their main use is in giving a model of an ∞-*category*. An ∞-category is meant to be a 'category' in which composition of morphisms is only well defined up to homotopy. There are several different models of ∞-categories and the two we shall discuss, Segal categories and quasi-categories, both use simplicial sets.

Let $\mathcal{C}$ be a small category. The *nerve* of $\mathcal{C}$, denoted $B\mathcal{C}$ is a simplicial set with $B\mathcal{C}_n = \{A_0 \stackrel{f_1}{\to} A_1 \stackrel{f_2}{\to} ... \stackrel{f_n}{\to} A_n\}$, the set of all strings of composable morphisms of length n. In other words, $B\mathcal{C}_n$ is the functor category $\mathcal{C}^{[n]}$, where $[n] = \{0, 1, ..., n\}$ is viewed as a category with morphisms given by the order on $[n]$. The face maps d_i are given by composing the morphisms at the ith place for $0 < i < n$ and removing the first (respectively last) morphism for $i = 0$ (respectively, $i = n$). The degeneracies s_i are given by inserting the identity at the ith place.

Since categories are (in some sense) easy, it is worthwhile knowing when a particular simplicial set is isomorphic to the nerve of a category. This question was answered by Segal in [Seg68], though the result is attributed to Grothendieck. The answer involves the Segal condition, which we now define.

Let $X : \Delta^{op} \to \mathbf{Set}$ be a simplicial set and consider the limit of the diagram $X_1 \overset{d_1}{\to} X_0 \overset{d_0}{\leftarrow} X_1 \overset{d_1}{\to} ... \overset{d_1}{\to} X_0 \overset{d_0}{\leftarrow} X_1$, hereby denoted $X_1 \times_{X_0} ... \times_{X_0} X_1$. For every n, there is a natural map $X_n \to X_1 \times_{X_0} ... \times_{X_0} X_1$ (where there are n factors), called the Segal map. This is induced by the maps $\alpha_i := X(\alpha^i)$, where $\alpha^i : [1] \to [n]$ is given by $0 \mapsto i$ and $1 \mapsto i + 1$.

Definition 1.2.3. A simplicial set $X : \Delta^{op} \to \mathbf{Set}$ satisfies the *Segal condition* if the Segal maps $X_n \to X_1 \times_{X_0} ... \times_{X_0} X_1$ are isomorphisms for all $n \in \mathbb{N}$.

One way of understanding the Segal condition is by viewing X_0 as a set of objects and X_1 as a set of morphisms connecting those objects, with d_1 giving the domain and d_0 the codomain. The Segal condition then says that every n-simplex can be represented uniquely as a string of compatible 1-simplices. In fact, this is exactly the proof of the Nerve theorem, which states that a simplicial set is the nerve of some category if and only if it satisfies the Segal condition.

Another answer to the question is given by the inner Kan condition. This is a somewhat more elementary condition to check than that given by the Segal condition. It is given by a filling condition similar to that of a fibration of simplicial sets using the horns Λ^n_k.

Definition 1.2.4. The *kth horn* of $\Delta[n]$ (where $\Delta[n] := \mathrm{Hom}_{\Delta}(-, [n])$), denoted by Λ^n_k, is the union of the images of d_i for $i \neq k$. A kth horn in a simplicial set X is a map of simplicial sets (i.e. a natural transformation) $\Lambda^n_k \to X$.

Considering the algebraic nature of simplicial sets it would be reasonable to expect a combinatorial description of a Λ^n_k horn inside a simplicial set X. This is given by the following lemma:

Lemma 1.2.5. Let X be a simplicial set. Then a Λ^n_k horn inside of X is exactly the data of n $(n-1)$-simplices $(\sigma_0, ..., \sigma_{k-1}, \sigma_{k+1}, ..., \sigma_n)$ such that for all $i, j \neq k$ we have $d_i\sigma_j = d_{j-1}\sigma_i$ if $i < j$.

Note that these relations are automatically satisfied by $(d_0\sigma, ..., d_{k-1}\sigma, d_{k+1}\sigma, ..., d_n\sigma)$ for any n-simplex σ. This means that we have a canonical map $H^n_k : X_n \to \Lambda^n_k(X)$, where $\Lambda^n_k(X)$ is the set of Λ^n_k-horns in X. We say that a Λ^n_k-horn can be filled if its pre-image under H^n_k is non-empty.

Using this description, we can analyse the horns of $B\mathcal{C}$ for a category $\mathcal{C}$. To start with, consider a Λ^2_1 horn. This is given by the data of two 1-simplices (morphisms) $f_0 : A \to B$ and $f_2 : C \to D$ such that $d_0 f_2 = d_1 f_0$. Note that for a 1-simplex $g : X \to Y$ we have that $d_0 g = X$ and $d_1 g = Y$. Thus from the above we have that (f_0, f_2) forms a Λ^2_1 horn if and only if $B = C$. In this case we can compose f_0 and f_1 to get a 2-simplex $A \stackrel{f_0}{\to} B \stackrel{f_2}{\to} D$ and so every Λ^2_1 horn in $B\mathcal{C}$ can be 'filled' to a Δ^2 simplex.

Next, let $(\sigma_0, \sigma_2, \sigma_3)$ be a Λ^3_1 horn. Write out $\sigma_i = A_i \stackrel{f_i}{\to} B_i \stackrel{g_i}{\to} C_i$. Then we know that $d_0\sigma_2 = d_1\sigma_0$, $d_0\sigma_3 = d_2\sigma_0$ and $d_2\sigma_3 = d_2\sigma_2$. What this translates to are the following equalities:

$$B_2 \stackrel{g_2}{\to} C_2 = A_0 \stackrel{g_0 \circ f_0}{\to} C_0, \quad B_3 \stackrel{g_3}{\to} C_3 = A_0 \stackrel{f_0}{\to} B_0, \quad A_3 \stackrel{f_3}{\to} B_3 = A_2 \stackrel{f_2}{\to} B_2$$

The most important equality here is the second. This says that the last morphism in σ_3 is equal to the first morphism in σ_0. This means that we can paste together these 2-simplices to get a 3-simplex. To be precise, since $C_3 = B_0$, we can add on $g_0 : B_0 \to C_0$ at the end of σ_3 to get the 3-simplex $A_3 \overset{f_3}{\to} B_3 \overset{g_3}{\to} C_3 = B_0 \overset{g_0}{\to} C_0$. It is easy to verify that this gives the horn $(\sigma_0, \sigma_2, \sigma_3)$. This leads us to the following:

Definition 1.2.6. A simplicial set X satisfies the *inner Kan condition* if given any $n, k \in \mathbb{N}$ with $0 < k < n$, H^n_k is bijective. That is, if every horn can be uniquely filled.

The above discussion can be generalised to show that the nerve of a category always satisfies the inner Kan condition. Another statement of the nerve theorem is that the converse also holds. In summary, a simplicial set X is the nerve of a category if and only if it satisfies the inner Kan condition. Collecting these statements we have the following:

Theorem 1.2.7. *Let $X : \Delta^{op} \to$ **Set** be a simplicial set. Then the following are equivalent:*

i) There exists a small category $\mathcal{C}$ such that $X \cong B\mathcal{C}$.

ii) X satisfies the Segal condition

iii) X satisfies the inner Kan condition

This theorem suggests that we can acquire models of ∞-categories by weakening conditions corresponding to the uniqueness of composition in categories. A *quasi-category* is a simplicial set X such that $H^n_k : X_n \to \Lambda^n_k(X)$ is surjective for all $n \geq 0$ and $k \in \mathbb{Z}$ with $0 < k < n$.

If we have a simplicial object in some model category we then get another model. A *segal category* is a simplicial object X such that the segal maps $X_n \to X_1 \times_{X_0} ... \times_{X_0} X_1$ are weak equivalences.

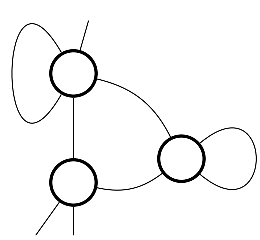
1.3 Topological graphs

In this thesis we use two different combinatorial definitions of graphs. The first description we give was introduced by Getzler and Kapranov in [GK98]. This description is included mostly for historical reasons as it was used in the original formulation of modular operads. The second combinatorial description, referred to as a Feynman graph, was introduced by Joyal and Kock in [JK11]. This description is better suited to describe the operation of graph substitution and provides a good model of ∞-modular operads.

However, our thinking in both of the combinatorial descriptions is guided by the intuitive notion of topological graphs. In this section we shall give a definition of a topological graph and in this setting introduce and give examples of terminology that will be used throughout this thesis. It is worth noting that while certain descriptions are better suited for certain situations, they are all equivalent.

Definition 1.3.1. A *topological graph* G is a pair (X, V), where X is a locally finite 1-dimensional CW-complex and $V \subseteq X$ is a finite collection of 0 cells of X such that $X \setminus V$ is a 1-manifold without boundary and with finitely many connected components.

Example 1.3.2. An example of a topological graph is



Here, there are three vertices, 8 edges and 3 legs.

A *vertex* of (X, V) is an element of V and an *edge* of X is a connected component of $X \setminus V$. The *exceptional graphs* are those such that $V = \emptyset$. An *arc* is an edge with a choice of orientation. Let A be the set of arcs of X. Then by swapping the chosen orientation we get a fixed-point free involution $i : A \to A$. Given an arc a we often use the notation $a^\dagger$ for the arc with the same underlying edge as a but with the opposite orientation.

Note that as $X \setminus V$ is a manifold without boundary, every connected component must be homeomorphic to S^1 or $(0, 1)$. Given an arc a homeomorphic to $(0, 1)$ we can find an embedding $f : (0, 1) \to X$ with image equal to the connected component given by a that induces the chosen orientation. We say that a is attached to the vertex $v \in V$ if $\lim_{t\to 1} f(t) = v$.

Note that as $X \setminus V$ is a manifold without boundary, $\lim_{t\to 1} f(t)$ must always be an element of V if it exists. This defines a partial function $t : A \not\to V$ and we denote the domain of t by D. Given a vertex v, we define the *neighbourhood* of v to be $nb(v) = t^{-1}(v)$. It is then clear that $D = \coprod_{v\in V} nb(v)$. If $t(a)$ does not exist, we call a a *leg* of G. The *boundary* of G, denoted $\partial(G)$, is the set of legs of G, that is $\partial(G) = A \setminus D$.

A *corolla* is a connected topological graph with a single vertex and no loops. That is, for any arc a exactly one of a or $a^\dagger$ is a leg. We denote the corolla with n legs by $*_n$. Given a topological graph X and a vertex $v \in V$ we can define the corolla $*_v$ as being the subcomplex of X with v the only 0-cell and all edges connected to v as its 1-cells.

Corollas are important because they generate all non-exceptional graphs. This means that if we associate an object in a category $\mathcal{C}$ to each corolla, we can use this information to assign an object to all graphs. This is the principle behind the definition of modular operads in Section 2. This idea is made precise by the following theorem:

Theorem 1.3.3. *The corollas generate all non-exceptional connected topo-*

logical graphs in the sense that every graph X which is connected and not homeomorphic to an exceptional edge or loop can be obtained by identifying legs of corollas.

1.4 Mapping class groups

Let S be a compact surface of genus g and n boundary components. All surfaces in this thesis are assumed to be orientable. There is an important group associated to S called the *mapping class group*. This is essentially the group of symmetries of S up to homotopy.

Explicitly, let $f : S \to S$ and $g : S \to S$ be homeomorphisms. We say that f is *isotopic* to g if there exists a homotopy $H : S \times I \to I$ such that $H_t(x)$ is a homeorphism of S for all $t \in I$. If this is the case, we write $f \simeq g$.

Now let $Homeo^+(S)$ be the group of orientation preserving homeomorphisms of S that fix the boundary pointwise. Then we have that the set $N = \{f \in Homeo^+(S) | f \simeq id_S\}$ is a normal subgroup of $Homeo^+(S)$. Hence we can define $\Gamma_{g,n} = Homeo^+(S)/N$. If S has punctures we can define the mapping class group in the same way and write $\Gamma_{g,n}^m$ if S has m punctures, n boundary components and genus g.

Definition 1.4.1. Let $\gamma : I \to S$ be a closed curve in S. The Dehn twist T_γ around γ is the equivalence class in $\Gamma_{g,n}$ of the homeomorphism defined by taking a tubular neighbourhood $\gamma(I) \times [-\epsilon, \epsilon]$ of γ and taking T_γ to be the identity outside of the tubular neighbourhood and $(x, t) \mapsto (se^{2\pi it}, t)$ on $\gamma(I) \times [-\epsilon, \epsilon] \cong S^1 \times [0, 2\pi i]$.

While this process does not define a unique homeomorphism it does define a unique equivalence class in $\Gamma_{g,n}$. The Dehn twists are special because they generate the mapping class groups.

Fundamental groups of surfaces fit into a short exact sequence involving mapping class groups. Specifically, if $S_{g,n}$ is a compact surface of genus g with n boundary components, the Birman exact sequence is given by

$$1 \longrightarrow \pi_1(S_{g,n}) \xrightarrow{Push} \Gamma^1_{g,n} \xrightarrow{\alpha} \Gamma_{g,n} \longrightarrow 1$$

where $\Gamma^1_{g,n}$ is the mapping class group of $S_{g,n} \setminus \{x\}$ for any $x \in S_{g,n}$. The map $Push$ is given by sending a homotopy class of curves $[\gamma]$ to the isotopy class of the homeomorphism of $S_{g,n} \setminus \{x\}$ given by 'pushing' the point x along a representative curve γ. The map α is simply given by forgetting the puncture.

This is proven using the long exact sequence of a fibration. Specifically, one considers the long exact sequence on homotopy groups corresponding to the fiber bundle $Homeo^+(S,x) \to Homeo^+(S) \to S$, where $Homeo^+(S)$ is the set of orientation preserving homeomorphisms of S with the compact-open topology and $Homeo^+(S,x)$ is the subspace of $Homeo^+(S)$ consisting of homeomorphisms fixing x. A good reference for the Birman exact sequence is [FM12]

1.5 Cyclic operads

In this section we outline the definition of an operad as given in [GK98]. This differs from the usual definition as it omits the unit axioms and the axiom regarding the point $* \in \mathcal{P}((0))$.

For the rest of this thesis Σ_n denotes the symmetric group on n letters. For simplicity we also assume that all categories in this thesis are concrete so that we can freely use elements.

Definition 1.5.1. An *operad* $\mathcal{P}$ in a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ consists of the following data:

- A collection of objects $\{\mathcal{P}((n))\}_{n\geq 0}$.

- An action of Σ_n on each $\mathcal{P}((n))$.
- For every $n, m \geq 0$ and all i satisfying $1 \leq i \leq n$ maps

$$\circ_i : \mathcal{P}((n)) \otimes \mathcal{P}((m)) \to \mathcal{P}((n+m-1)).$$

This data needs to satisfy the following axioms:

i) The Σ_n action is compatible with the maps $\circ_i$ in the following sense: Given $\sigma \in \Sigma_n$, $\pi \in \Sigma_m$, $a \in \mathcal{P}((n))$ and $b \in \mathcal{P}((m))$ we have

$$(\sigma a) \circ_{\sigma(i)} (\pi b) = (\sigma \circ_i \pi)(a \circ_i b)$$

Here $\sigma \circ_i \pi$ is the permutation defined by

$$(\sigma \circ_i \pi)(j) = \begin{cases} \sigma(j) & 1 \leq j < i \text{ and } \sigma(j) < \sigma(i) \\ \sigma(j) + m - 1 & 1 \leq j < i \text{ and } \sigma(j) > \sigma(i) \\ \sigma(i) + \pi(j - i + 1) - 1 & i \leq j < i + m \\ \sigma(j - m + 1) & i + m \leq j \leq n + m \text{ and } \sigma(j) < \sigma(i) \\ \sigma(j - m + 1) + m - 1 & i + m \leq j \leq n + m \text{ and } \sigma(j) > \sigma(i) \end{cases}$$

ii) For $a \in \mathcal{P}((k))$, $b \in \mathcal{P}((l))$ and $c \in \mathcal{P}((m))$ and $1 \leq i, j < k$ we have

$$(a \circ_i b) \circ_{j+l-1} c = (a \circ_j c) \circ_i b$$

iii) For a, b, c as before and $1 \leq i \leq k$, $1 \leq j \leq l$ we have

$$(a \circ_i b) \circ_{i+j-1} c = a \circ_i (b \circ_j c).$$

Recall that given a simplicial set X the elements of X_n can be viewed as graphs formed by a straight line with n vertices and $n+1$ edges (see Example 1.2.2). The face maps and degeneracies are then represented by collapsing and adding edges respectively. There is a similar interpretation of operads; an element of $\mathcal{P}((n))$ is a rooted tree with n inputs and 1 output.

If we label these trees so that the root is labelled 0 there is a natural Σ_n action on the set of labelled rooted trees given by permuting the non-zero labels. Given a rooted tree T_1 with n inputs and another rooted tree T_2 with m inputs we can combine them to form a rooted tree $T_1 \circ_i T_2$ with $n+m-1$ inputs by grafting the root of T_2 to the ith input of T_1.

The axioms for an operad just require that the operadic compositions $\circ_i$ behave similarly to this grafting operation with respect to the Σ_n action.

Definition 1.5.2. A *cyclic operad* $\mathcal{P}$ consists of the following data:

- An operad $\mathcal{P}$
- An action of $\Sigma_{n+} = Aut\{0, ..., n\}$ on $\mathcal{P}((n))$

Satisfying the following axiom: For any $a \in \mathcal{P}((n))$ and $b \in \mathcal{P}((m))$, we have $\tau(a \circ_n b) = (\tau b) \circ_1 (\tau a)$, where $\tau = (01...n) \in \Sigma_{n+}$

Cyclic operads can be represented by non-rooted trees. Note that we can define operations $\circ_{ij} : \mathcal{P}((n)) \otimes \mathcal{P}((m)) \to \mathcal{P}((n+m-1))$ by $a \circ_{ij} b = a \circ_i \tau^{-j} b$. This can be thought of as attaching the jth input of T_2 to the ith input of T_1.

Chapter 2

Modular Operads

In this chapter we will give a review of the definition of modular operads as given by Getzler and Kapranov in [GK98] and provide proofs of two important characterizations. These characterizations of modular operads will provide a link to the presheaf definition used later in this thesis.

This chapter largely follows [GK98].

2.1 Getzler and Kapranov's graphical category

Here we describe a category of graphs Γ with full subcategories $\Gamma((g, n))$ for each $g, n \geq 0$.

Definition 2.1.1. A *graph* consists of the following data:

- A finite set $\mathrm{Flag}(G)$
- An involution $i : \mathrm{Flag}(G) \to \mathrm{Flag}(G)$
- A partition of $\mathrm{Flag}(G)$ λ.

The blocks of λ are called *vertices* and the set of equivalence classes is denoted by $\mathrm{Vert}(G)$. A flag a meets the vertex v if $a \in v$. If a is a flag with

$i(a) \neq a$, we call the set $\{a, i(a)\}$ an *edge*. If $a = i(a)$ we call a a *leg*. The *valence* of a vertex v, denoted $n(v)$, is $|v|$.

Definition 2.1.2. A *morphism of graphs* $f : G_0 \to G_1$ is given by an injection $f^* : \mathrm{Flag}(G_1) \to \mathrm{Flag}(G_0)$ satisfying the following conditions:

- $i_0 f^* = f^* i_1$, where i_j is the involution in G_j.
- i_0 acts freely on the complement of the image of f^*. Equivalently, f^* is surjective (and thus bijective) on legs.
- Two flags a and b in G_1 meet if, and only if, there is a sequence $(x_0, ..., x_k)$ of flags in G_0 such that $f^*(a) = x_0$, $i_0(x_{j-1})$ meets x_j for all $1 \leq j \leq k$ and $f^*(x_k) = b$.

Given a vertex $v \in \mathrm{Vert}(G_1)$, $f^{-1}(v)$ denotes the subgraph of G_1 given by all flags a that are connected to a leg of v by a sequence of flags in the complement of the image of f^*.

This definition can be seen as saying that a morphism $f : G_0 \to G_1$ is given by contracting subgraphs of G_0. Specifically, G_1 is obtained by contracting all edges in the complement of the image of f^*.

Given a graph G we can construct a topological graph $|G|$ in the following way: Take a copy of $[0, \frac{1}{2}]$ for each flag in G and identify $0 \in [0, \frac{1}{2}]$ for all copies corresponding to flags in the same equivalence class. Then identify $\frac{1}{2} \in [0, \frac{1}{2}]$ in any pair of copies corresponding to flags in the same orbit under i.

Definition 2.1.3. A *labelled graph* is a graph G along with a function $g : \mathrm{Vert}(G) \to \mathbb{Z}_{\geq 0}$. A *stable graph* is a labelled graph satisfying the extra condition that $2(g(v)-1)+n(v) > 0$ for all vertices $v \in \mathrm{Vert}(G)$. A *morphism of stable graphs* is a morphism of graphs such that $g(v) = g(f^{-1}(v))$ for all $v \in \mathrm{Vert}(G_1)$.

The *genus* of a stable graph G is defined to be

$$g(G) = \sum_{v \in \mathrm{Vert}(G)} g(v) + \beta_1(|G|),$$

where $\beta_1(X) = \dim_{\mathbb{Q}}(H_1(X))$ is the first betti number of X. As

$$\beta_1(|G|) = |\mathrm{Edge}(G)| - |\mathrm{Vert}(G)| + 1,$$

we can equivalently write this as

$$g(G) = \sum_{v \in \mathrm{Vert}(G)} (g(v) - 1) + |\mathrm{Edge}(G)| + 1.$$

Theorem 2.1.4. *There is a category Γ' whose objects are stable graphs and whose morphisms are morphisms of stable graphs.*

Proof. First, given $f : G_0 \to G_1$ and $h : G_1 \to G_2$, let $h \circ f : G_0 \to G_2$ to be given by f^*h^*. We need to show that this composition is a morphism of stable graphs.

First, we have $i_0 f^* h^* = f^* i_1 h^* = f^* h^* i_2$ as required. Next, note that as both f^* and h^* are bijective on legs, so is f^*h^*. Now, suppose that two flags a and b meet in G_2. Then there exist a sequence $(x_0, ..., x_k)$ of flags in G_1 such that $h^*(a) = x_0$, $i_1(x_{j-1})$ meets x_j for all $1 \leq j \leq k$ and $h^*(x_k) = b$. As $i_1(x_{j-1})$ meets x_j, there is a sequence $(y^j_0, ..., y^j_{n_j})$ satisfying similar conditions with f^*. Then $(y^0_0, ..., y^0_{n_0}, y^1_0, ..., y^k_{n_k})$ is a sequence of flags in G_0 satisfying the required conditions for f^*h^*.

Thus composition of morphisms of graphs is well defined. That it is associative follows from associativity of function composition. The identity id_G is then given by $id_{\mathrm{Flag}(G)}$. □

The category Γ' is not quite sufficient for our purposes, so we enlarge it by ordering the legs of all graphs.

Definition 2.1.5. The category Γ has pairs (G, ρ) as its objects, where ρ is an ordering of the legs of G. Here an *ordering* is a bijection $\rho : \mathbb{Z}/n\mathbb{Z} \to Leg(G)$. The morphisms in this category are morphisms of stable graphs that preserve the ordering. There is a full subcategory $\Gamma((g, n))$ whose objects are the graphs with genus g and n legs.

That Γ is a category follows simply from Theorem 2.1.4. Note that $\Gamma((g, n))$ has a terminal object, denoted $*_{g,n}$, called the corolla of genus g with n legs. The corolla $*_{g,n}$ consists of a single vertex v with $g(v) = g$ and n legs with no edges.

Lemma 2.1.6. There are finitely many isomorphism classes in $\Gamma((g, n))$ for all $g, n \geq 0$.

Proof. Let $G \in \Gamma((g, n))$ be any graph. Since every flag in G is either a leg or part of an edge, we can see that $\sum_{v \in V} n(v) = 2|Edge(G)| + n$. Subtracting this formula from $3(g - 1) = 3\sum_{v \in V}(g(v) - 1) + 3|Edge(G)|$ we can see that

$$3(g-1) + n = |Edge(G)| + \sum_{v \in Vert(G)} \big[3(g(v) - 1) + n(v)\big]$$

Since G is stable, $\sum_{v \in Vert(G)}[3(g(v) - 1) + n(v)] \geq 0$. This gives that $|Edge(G)| \leq 3(g - 1) + n$ which in turn shows

$$|Flag(G)| \leq 6(g - 1) + 2n.$$

There are clearly only finitely many isomorphism classes of graphs with a particular number of flags, and so there can only be finitely many isomorphism classes in $\Gamma((g, n))$ as required. □

2.2 Modular operads

Let $(\mathcal{C}, \otimes, I)$ be a distributive symmetric monoidal category with all finite colimits. For example, the category of k-vector spaces k-**Vec** with the tensor product or the category of topological spaces **Top** with the cartesian product.

A *stable* Σ*-module* in $\mathcal{C}$ is a collection of objects

$$\{\mathcal{V}((g,n))|g,n \geq 0, 2g-2+n>0\},$$

each of which has an action of the symmetric group Σ_n.

There is a category of stable Σ-modules, where morphisms are given by collections of equivariant maps $\mathcal{V}((g,n)) \to \mathcal{W}((g,n))$. Given a finite set I with $n = |I|$, let

$$\mathcal{V}((g,I)) = \Big(\coprod_{\mathrm{Iso}(\{1,\dots,n\},I)} \mathcal{V}((g,n)) \Big)_{\Sigma_n}$$

Note that this object has a natural $Aut(I)$ action.

One can think of $\mathcal{V}((g,n))$ as attaching an object in $\mathcal{C}$ to the corolla $*_{g,n}$. Using the tensor product in $\mathcal{C}$, we can naturally extend this assignment to all graphs G as follows:

Definition 2.2.1. Let G be a stable graph and $\mathcal{V}$ a stable Σ-module. Recalling that a vertex v of G is a finite subset of $\mathrm{Flag}(G)$, define

$$\mathcal{V}((G)) = \bigotimes_{v \in \mathrm{Vert}(G)} \mathcal{V}((g(v),v))$$

Theorem 2.2.2. *There is a monad* $\mathbb{M}$ *on the category of stable* Σ*-modules given on objects by*

$$\mathbb{M}\mathcal{V}((g,n)) = \underrightarrow{\mathrm{colim}}_{G \in Iso(\Gamma((g,n)))} \mathcal{V}((G)) \cong \coprod_{G \in [\Gamma((g,n))]} \mathcal{V}((G))_{Aut(G)}$$

Where $[\Gamma((g,n))]$ *is the set of isomorphism classes in* $\Gamma((g,n))$*.*

Proof. First note that this monad is well defined on objects as by Lemma 2.1.6 it is a finite colimit.

We are viewing $\mathcal{V}$ as a functor from $Iso(\Gamma((g,n))) \to \mathcal{C}$. We can do this as an isomorphism of graphs can only exchange edges attached to the same vertex that together form part of a loop. Hence this is just an automorphism of v for every $v \in Vert(G)$ and using the tensor product together with the $Aut(v)$ action on $\mathcal{V}((*_v))$ we can get a morphism on $\mathcal{V}((G))$.

Now, let $f : \mathcal{V} \to \mathcal{V}'$ be a morphism of stable Σ-modules. That is, f is a collection of Σ_n equivariant maps $f_{g,n} : \mathcal{V}((g,n)) \to \mathcal{V}'((g,n))$. We need to show that there exists morphisms $\mathbb{M}f_{g,n} : \mathbb{M}\mathcal{V}((g,n)) \to \mathbb{M}\mathcal{V}'((g,n))$. This follows from the more general statement that if $\eta : F \implies G$ is a natural transformation of functors such that $\varinjlim F$ and $\varinjlim G$ both exist, then there is a morphism $\varinjlim \eta : \varinjlim F \to \varinjlim G$.

To see this, let $\phi_X : G(X) \to \varinjlim G$ be the morphisms making $\varinjlim G$ into a cocone over G. Then we claim that $\psi_X = \phi_X \eta_X$ make $\varinjlim G$ into a cocone over F so that by universality of $\varinjlim F$ there is a unique morphism $\varinjlim \eta : \varinjlim F \to \varinjlim G$. This is a simple calculation:

$$\begin{aligned}
\psi_Y F(f) &= \phi_Y \eta_Y F(f) \\
&= \phi_Y G(f) \eta_X \\
&= \phi_X \eta_X \\
&= \psi_X
\end{aligned}$$

That $\varinjlim \eta\mu = \varinjlim \eta \varinjlim \mu$ and $\varinjlim id_F = id_{\varinjlim F}$ follows from uniqueness.

Now, to show that $\mathbb{M}$ is a monad we need to construct natural transformations $\eta : id \to \mathbb{M}$ and $\mu : \mathbb{M}\mathbb{M} \to \mathbb{M}$. To define η we need to give a morphism $\eta_{g,n} : \mathcal{V}((g,n)) \to \varinjlim_{G \in [\Gamma((g,n))} \mathcal{V}((G))$ for all $g, n \geq 0$. To do this, we can simply take the composition

$$\mathcal{V}((g,n)) \overset{\cong}{\to} \mathcal{V}((*_{g,n})) \to \varinjlim_{G \in [\Gamma((g,n))]} \mathcal{V}((G))$$

Now we need to construct μ. Note that

$$\mathbb{M}\mathcal{V}((G)) = \otimes_{v \in Vert(G)} \coprod_{G \in Iso(\Gamma((g(v),n(v))))} \mathcal{V}((G))_{Aut(G)} \cong \coprod_{I \subseteq Edge(G)} \mathcal{V}((G))_{Aut(G)}$$

Recall that morphisms in $\Gamma((g,n))$ are given (up to isomorphism) by contracting edges. This gives

$$\mathbb{M}^2\mathcal{V}((g,n)) \cong \varinjlim_{G_0 \to G_1 \in Iso_2\Gamma((g,n))} \mathcal{V}((G_0))$$

Where $Iso_2\Gamma((g,n))$ is the category with objects diagrams of the form $G_0 \to G_1$ and morphisms isomorphisms of diagrams. The summand given by $f : G_0 \to G_1$ is $\bigotimes_{w \in Vert(G_1)} \mathcal{A}((f^{-1}(w))) \cong \mathcal{A}((G_0))$. By sending the summand in $\mathbb{M}^2\mathcal{V}((g,n))$ corresponding to $G_0 \to G_1$ to the summand corresponding to G_0 in $\mathbb{M}\mathcal{V}((g,n))$ we get a natural transformation $\eta : \mathbb{M}^2 \to \mathbb{M}$.

The details checking that these natural transformations satisfy the axioms of a monad can be found in [GK98] □

Definition 2.2.3. A *modular operad* is an algebra over the monad $\mathbb{M}$. That is, it is a stable Σ-module $\mathcal{A}$ along with a structure map $h : \mathbb{M}\mathcal{A} \to \mathcal{A}$ such that the diagrams

$$\begin{array}{ccc} \mathbb{M}^2\mathcal{A} & \xrightarrow{\mathbb{M}h} & \mathbb{M}\mathcal{A} \\ \downarrow{\scriptstyle \mu_{\mathcal{A}}} & & \downarrow{\scriptstyle h} \\ \mathbb{M}\mathcal{A} & \xrightarrow{h} & \mathcal{A} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\eta_{\mathcal{A}}} & \mathbb{M}\mathcal{A} \\ & \searrow^{id_{\mathcal{A}}} & \downarrow{\scriptstyle h} \\ & & \mathcal{A} \end{array}$$

both commute.

A *morphism of modular operads* $f : \mathcal{A} \to \mathcal{A}'$ is a morphism of stable Σ-modules such that

$$\begin{array}{ccc} \mathbb{M}\mathcal{A} & \xrightarrow{\mathbb{M}f} & \mathbb{M}\mathcal{A}' \\ \downarrow{\scriptstyle h_{\mathcal{A}}} & & \downarrow{\scriptstyle h_{\mathcal{A}'}} \\ \mathcal{A} & \xrightarrow{f} & \mathcal{A}' \end{array}$$

commutes

Theorem 2.2.4. *Let $(\mathcal{C}, \otimes, I)$ be a distributive symmetric monoidal category with all finite colimits. Then the modular operads in $\mathcal{C}$ form a category, which we denote by **Mod$\mathcal{C}$**.*

Proof. Let $f : \mathcal{A} \to \mathcal{A}'$ and $g : \mathcal{A}' \to \mathcal{A}''$ be morphisms of modular operads. Note that commutativity of the diagram

$$\begin{array}{ccc} \mathbb{M}\mathcal{A} & \xrightarrow{\mathbb{M}(gf)} & \mathbb{M}\mathcal{A}'' \\ \downarrow{\scriptstyle h_{\mathcal{A}}} & & \downarrow{\scriptstyle h_{\mathcal{A}''}} \\ \mathcal{A} & \xrightarrow{gf} & \mathcal{A}'' \end{array}$$

is equivalent to commutativity of

$$\begin{array}{ccccc} \mathbb{M}\mathcal{A} & \xrightarrow{\mathbb{M}f} & \mathbb{M}\mathcal{A}' & \xrightarrow{\mathbb{M}g} & \mathbb{M}\mathcal{A}'' \\ \downarrow{\scriptstyle h_{\mathcal{A}}} & & \downarrow{\scriptstyle h_{\mathcal{A}'}} & & \downarrow{\scriptstyle h_{\mathcal{A}''}} \\ \mathcal{A} & \xrightarrow{f} & \mathcal{A}' & \xrightarrow{g} & \mathcal{A}'' \end{array}$$

By functoriality of $\mathbb{M}$. This then commutes as both f and g are morphisms of modular operads. Hence composition of morphisms of modular operads is well defined.

Next, as $\mathbb{M}id_{\mathcal{A}} = id_{\mathbb{M}\mathcal{A}}$ it is clear that the identity is a morphism of modular operads. Finally, associativity follows from associativity of morphisms of stable Σ-modules, which in turn follows from associativity of morphisms in $\mathcal{C}$.

□

Example 2.2.5. Let V be a chain complex with a symmetric inner product

$B(x,y)$ satisfying $B(x,y) = 0$ if $\deg x + \deg y \neq 0$. The endomorphism modular operad of V is defined by $\mathcal{E}[V]((g,n)) = V^{\otimes n}$. The action of Σ_n is given by permuting factors. We then have

$$\mathcal{E}[V]((G)) \cong \bigotimes_{v \in Vert(G)} V^{\otimes |v|} \cong V^{\otimes (\sum_{v \in Vert(G)} |v|)} \cong V^{\otimes Flag(G)}$$

In particular, $\mathcal{E}[V]((*_{g,n})) \cong \mathcal{E}[V]((g,n))$. Thus we can define a morphism $B^{\otimes Edge(G)} : \mathcal{E}[V]((G)) \to \mathcal{E}[V]((g,n))$ by applying B to factors corresponding to a pair of flags in the same edge. By the universal property of a colimit, this gives a structure map $\mathbb{M}\mathcal{E}[V] \to \mathcal{E}[V]$. It can be shown that this structure map does indeed make $\mathcal{E}[V]$ into a modular operad.

2.3 Modular operads as functors

Given a stable Σ-module $\mathcal{A}$ we can define a 'functor' $\Gamma \to \mathcal{C}$ on objects as in Definition 2.2.1. It is natural to ask when this can be extended to a true functor from Γ to $\mathcal{C}$. The results in this section show that we can do this exactly when $\mathcal{A}$ is a modular operad.

To be precise, a modular pre-operad is a stable Σ-module $\mathcal{A}$ together with a structure map $h : \mathbb{M}\mathcal{A} \to \mathcal{A}$. Given a modular pre-operad and a graph $G \in \Gamma((g,n))$, let h_G be the composition $\mathcal{A}((G)) \to \mathbb{M}\mathcal{A}((g,n)) \xrightarrow{h} \mathcal{A}((g,n))$.

Now, given a map $f : G_0 \to G_1$, define $\mathcal{A}((f))$ to be the composition

$$\mathcal{A}((G_0)) \cong \bigotimes_{v \in \mathrm{Vert}(G_1)} \mathcal{A}((f^{-1}(v))) \xrightarrow{\otimes_v h_{f^{-1}(v)}} \mathcal{A}((G_1))$$

Lemma 2.3.1. A modular pre-operad $\mathcal{A}$ is a modular operad if and only if $\mathcal{A}((id_{*_{g,n}})) = id_{A((*_{g,n}))}$ and $\mathcal{A}((fg)) = \mathcal{A}((f))\mathcal{A}((g))$ for all maps of the form $G_0 \xrightarrow{f} G_1 \xrightarrow{g} *_{g,n}$.

Proof. First, note that $id_{\mathcal{A}((g,n))} = \mathcal{A}((id_{*_{g,n}})) = h\eta_{g,n}$ as $\eta_{g,n}$ is by definition the map $\mathcal{A}((g,n)) \cong \mathcal{A}((*_{g,n})) \to \mathbb{M}\mathcal{A}((g,n))$. Thus h satisfies the unital axiom of a monad.

Next note that the required condition is equivalent to commutativity of the following diagram for all $f : G_0 \to G_1$

$$\begin{array}{ccc} \mathcal{A}((G_0)) & \xrightarrow{\mathcal{A}((f))} & \mathcal{A}((G_1)) \\ \downarrow h_{G_0} & \swarrow h_{G_1} & \\ \mathcal{A}((g,n)) & & \end{array}$$

To see associativity, recall that a summand in $\mathbb{M}\mathcal{A}((g,n))$ corresponds to a morphism $f : G_0 \to G_1$ and that this summand is actually $\bigotimes_{w\in Vert(G_1)} \mathcal{A}((f^{-1}(w)))$. Then the map $\mathbb{M}h$ is given by sending this summand to $\mathcal{A}((G_1))$ by collapsing each $f^{-1}(w)$ using h. This is just the map $\mathcal{A}((f))$. Then by commutativity of the above diagram we can see that h satisfies the associativity axiom on all summands of $\mathbb{M}\mathcal{A}((g,n))$ and thus satisfies it for $\mathbb{M}\mathcal{A}$ itself. Hence $\mathcal{A}$ is a modular operad.

Conversely, by an entirely similar argument we see that if $\mathcal{A}$ is a modular operad then $\mathcal{A}((id_{*_{g,n}})) = id_{\mathcal{A}((*_{g,n}))}$ and that the above diagram commutes. □

We then have the following characterization of modular operads:

Theorem 2.3.2. *A modular pre-operad is a modular operad if, and only if, it induces a functor $\mathcal{A} : \Gamma \to \mathcal{C}$.*

Proof. By Lemma 2.3.1 if $\mathcal{A}$ induces a functor it is also a modular operad.

Conversely, let $\mathcal{A}$ be a modular operad and $f : G_0 \to G_1$ and $g : G_1 \to G_2$ be any morphisms. Then again by Lemma 2.3.1 we have

$$\mathcal{A}(((gf)^{-1}(w) \to f^{-1}(w) \to *_w)) = \mathcal{A}((f^{-1}(w) \to *_w))\mathcal{A}(((gf)^{-1}(w) \to f^{-1}(w)))$$

for all $w \in Vert(G_2)$. Hence the composition $\mathcal{A}((g))\mathcal{A}((f))$ can be written as

$$\mathcal{A}((G_0)) \cong \bigotimes_{w \in Vert(G_2)} \mathcal{A}(((gf)^{-1}(w))) \xrightarrow{\otimes h_{(gf)^{-1}(w)}} \mathcal{A}((G_2))$$

which is exactly $\mathcal{A}((gf))$. □

Corollary 2.3.3. Let $\mathcal{A}$ and $\mathcal{A}'$ be modular operads. A morphism of stable Σ-modules $\eta : \mathcal{A} \to \mathcal{A}'$ is a morphism of modular operads if, and only if, $\eta_G = \bigotimes_{v \in V} f_{g(v),v}$ gives a natural transformation of functors.

Proof. The naturality diagram for η is

$$\begin{array}{ccc} \mathcal{A}((G_0)) & \xrightarrow{\eta_{G_0}} & \mathcal{A}'((G_0)) \\ \downarrow{\scriptstyle \mathcal{A}((f))} & & \downarrow{\scriptstyle \mathcal{A}'((f))} \\ \mathcal{A}((G_1)) & \xrightarrow{\eta_{G_1}} & \mathcal{A}'((G_1)) \end{array}$$

where $f : G_0 \to G_1$ is any morphism. Note that

$$\eta_{G_1} \circ \mathcal{A}((f)) = \bigotimes_{v \in Vert(G_1)} \eta_{*_v} \circ h_{f^{-1}(v)}$$

and

$$\mathcal{A}'((f)) \circ \eta_{G_0} = \bigotimes_{v \in Vert(G_1)} h'_{f^{-1}(v)} \circ \eta_{f^{-1}(v)}$$

Hence commutativity of the naturality diagram is equivalent to the commutativity of

$$\begin{array}{ccc} \mathcal{A}((G)) & \xrightarrow{\eta_{G}} & \mathcal{A}'((G)) \\ \downarrow{\scriptstyle h_G} & & \downarrow{\scriptstyle h'_G} \\ \mathcal{A}((*_{g,n})) & \xrightarrow{\eta_{*_{g,n}}} & \mathcal{A}'((*_{g,n})) \end{array}$$

for all $G \in \Gamma((g,n))$. Noting that h is the sum of h_G over all $G \in \Gamma((g,n))$ we can see that this is in fact equivalent to commutativity of

$$\begin{array}{ccc} \mathbb{M}\mathcal{A} & \xrightarrow{\mathbb{M}\eta} & \mathbb{M}\mathcal{A}' \\ \downarrow h & & \downarrow h' \\ \mathcal{A} & \xrightarrow{\eta} & \mathcal{A}' \end{array}$$

as required. □

2.4 Modular operads as cyclic operads with extra structure

Here we show that modular operads are cyclic operads with extra structure; a grading given by genus and 'contraction maps' that increase genus.

Definition 2.4.1. A *stable graded cyclic operad* is a cyclic operad $\mathcal{P}$ with a Σ_n-invariant decomposition

$$\mathcal{P}((n)) = \coprod_{\substack{g\geq 0 \\ 2(g-1)+n>0}} \mathcal{P}((g, n+1))$$

where $\mathcal{P}((g,n))$ is a stable Σ-module such that the action of Σ_n on $\mathcal{P}((n))$ is induced by the Σ_{n+} action on $\mathcal{P}((g,n+1))$. We also require that for all $g,h,n,m \geq 0$ with $2(g-1)+n > 0$ and $2(h-1)+n > 0$, there is a map $\circ_i^{g,h} : \mathcal{A}((g,n)) \otimes \mathcal{A}((h,m)) \to \mathcal{A}((g+h, n+m-2))$ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{A}((g,n)) \otimes \mathcal{A}((h,m)) & \hookrightarrow & \mathcal{A}((n-1)) \otimes \mathcal{A}((m-1)) \\ \downarrow \circ_i^{g,h} & & \downarrow \circ_i \\ \mathcal{A}((g+h, n+m-2)) & \hookrightarrow & \mathcal{A}((n+m-3)) \end{array}$$

Remark. The shifting to $n+1$ in the above decomposition is because $\mathcal{P}((n))$ should represent operations with n inputs and 1 output. Whereas $\mathcal{P}((g,n))$ is representing a corolla with n legs. As such, when talking about labelled graphs in the context of stable graded cyclic operads we consider the ordering as starting at 0 and ending at n.

Also, note that given a stable graded cyclic operad $\mathcal{P}$, $\mathcal{P}_0((n)) = \mathcal{P}((0, n+1))$ inherits the composition maps from $\mathcal{P}$ and is a cyclic sub-operad.

Lemma 2.4.2. Let $\mathcal{A}$ be a modular operad. Then $\mathcal{A}((n)) = \coprod_{g\geq 0} \mathcal{A}((g, n+1))$ is a stable graded cyclic operad.

Proof. We need to construct maps $\circ_i : \mathcal{A}((n)) \otimes \mathcal{A}((m)) \to \mathcal{A}((n+m-1))$. As $\mathcal{C}$ is distributive, $\mathcal{A}((n)) \otimes \mathcal{A}((m)) \cong \coprod_{g,h\geq 0} \mathcal{A}((g, n+1)) \otimes \mathcal{A}((h, m+1))$. Let G be the graph given by attaching the 0th leg of $*_{h,m+1}$ to the ith leg of $*_{g,n+1}$. Note that $\mathcal{A}((G)) \cong \mathcal{A}((g, n+1)) \otimes \mathcal{A}((h, m+1))$.

We can define $\circ_i^{g,h} = \mathcal{A}((f))$ where $f : G \to G/I$ is the map given by collapsing the internal edge of G. Note that $G/I \cong *_{g+h,n+m}$ so that the codomain of $\circ_i^{g,h}$ is $\mathcal{A}((g+h, n+m))$. We can compose $\circ_i^{g,h}$ with the inclusions $\mathcal{A}((g+h, n+m)) \hookrightarrow \coprod_{g'\geq 0} \mathcal{A}((g', n+m)) = \mathcal{A}((n+m-1))$ to get maps $\mathcal{A}((g, n+1)) \otimes \mathcal{A}((h, m+1)) \to \mathcal{A}((n+m-1))$. Summing over all $g, h \geq 0$ we get maps $\circ_i : \mathcal{A}((n)) \otimes \mathcal{A}((m)) \to \mathcal{A}((n+m-1))$.

These maps satisfy the axioms of Definition 1.5.2 because the corresponding operations on graphs satisfy them. □

Corollary 2.4.3. Let $\mathcal{A}$ be a modular operad. Then $\mathcal{A}_0((n)) = \mathcal{A}((0, n+1))$ is a cyclic operad

Next we have the last characterisation of modular operads.

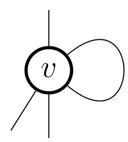
Theorem 2.4.4. *Let $\mathcal{A}$ be a graded cyclic operad with contraction maps $\xi_{i,j} : \mathcal{A}((g, I)) \to \mathcal{A}((g+1, I \setminus \{i, j\}))$ satisfying $\xi_{\sigma(i),\sigma(j)} \cdot \sigma = \sigma \cdot \xi_{i,j}$ (henceforth referred to as equivariance) for all $\sigma \in \Sigma_n$. Then this data gives a modular operad if, and only if, the following relations hold:*

- *For any finite set I and distinct elements $i, j, k, l \in I$ we have $\xi_{i,j} \circ \xi_{k,l} = \xi_{k,l} \circ \xi_{i,j}$*

- *If* $a \in \mathcal{A}((n))$ *and* $b \in \mathcal{A}((m))$ *then* $\xi_{1,2}(a \circ_n b) = (\xi_{1,2}a) \circ_n b$
- *With* a, b *as before, we have* $\xi_{n,n+1}(a \circ_n b) = a \circ_n (\xi_{1,2}b)$
- *With* a, b *as before, we have* $\xi_{n-1,n}(a \circ_n b) = \xi_{n+m-2,n+m-1}(a \circ_{n-1} \tau^{-1}b)$ *where* $\tau = (01...n) \in \Sigma_{n+}$.

Proof. For the forwards direction, by Lemma 2.4.2 we already have a stable graded cyclic operad. Hence we just need to construct the contraction maps $\xi_{i,j}$ and verify that they satisfy the above relations.

To do this consider the graph G with one vertex v of genus g, set of flags I and one edge e formed by the two flags $i, j \in I$. This is a contracted corolla and an example is shown below



We then have $\mathcal{A}((G)) \cong \mathcal{A}((g, I))$ where I is the set of flags in G. Let $f : G \to G/e$ be the map given by contracting the one edge e of G. Since we have $\mathcal{A}((G/e)) \cong \mathcal{A}((g+1, I \setminus \{i,j)) \}$ we can define $\xi_{i,j} = \mathcal{A}((f)) : \mathcal{A}((g, I)) \to \mathcal{A}((g+1, I \setminus \{i,j\}))$.

The fact that $\xi_{i,j}$ as defined obeys the relevant relations follows from the fact that the corresponding operations on graphs obey these relations.

Conversely, suppose that $\mathcal{A}$ is a stable graded cyclic operad with contraction maps $\xi_{i,j} : \mathcal{A}((g, I)) \to \mathcal{A}((g+1, I \setminus \{i,j\}))$ satisfying the above relations. We shall use this information to define a functor $\mathcal{A} : \Gamma \to \mathcal{C}$. Given such a functor, the maps $G \to *_{g,n}$ induce a morphism $h : \mathbb{M}\mathcal{A} \to \mathcal{A}$ and by Theorem 2.3.2 this will make $\mathcal{A}$ into a modular operad.

First, we define $\mathcal{A}$ on objects by the usual formula and on isomorphisms using the same method as in the proof of Theorem 2.2.2. Recall that every morphism in Γ is a series of contractions followed by an isomorphism. Hence we only need to define how $\mathcal{A}$ acts on contractions of single edges.

We have two cases to consider. First we consider the case $f : G \to G/e$ where the edge e has distinct endpoints v and v'. Then $\mathcal{A}((G)) \cong \mathcal{A}((H)) \otimes \bigotimes_{w \neq v,v'} \mathcal{A}((g(w), w))$ where H is the subgraph of G with vertices v and v', single edge e and legs the legs of v and v'. H/e has one vertex $\overline{v}$ and we have $\mathcal{A}((G/e)) \cong \mathcal{A}((g(\overline{v}), \overline{v})) \otimes \bigotimes_{w \neq v,v'} \mathcal{A}((g(w), w))$. Noting that $n(\overline{v}) = n(v) + n(v') - 2$ we can define $\mathcal{A}((f)) = \circ_{i,j}^{g,h} \otimes \bigotimes_{w \neq v,v'} id$, where i and j are the legs of v and v' respectively that make up e.

Next we consider the case $f : G \to G/e$ where the edge e has just one endpoint v. Similarly to the first case, if i and j are the flags making up e, we can define $\mathcal{A}((f)) = \xi_{i,j} \otimes \bigotimes_{w \neq v} id$.

To show that this gives a well defined functor $\mathcal{A} : \Gamma \to \mathcal{C}$ we need to show that if e_1 and e_2 are two edges in G then $\mathcal{A}((G \to G/e_1 \to (G/e_1)/e_2)) = \mathcal{A}((G \to G/e_2 \to (G/e_2)/e_1))$. If e_1 and e_2 do not meet then the result is clear. If they do meet, well definedness can be checked by examining the four possible ways a graph can have two internal edges and using the relations to check that the result is independent of our choice. More details can be found in [GK98]

Finally, as mentioned above, this gives a map $h : \mathbb{M}\mathcal{A} \to \mathcal{A}$. Specifically, h is defined by summing $\mathcal{A}((G \to *_{g,n}))$ over all $G \in \Gamma((g, n))$. By construction the pre-modular operad $(\mathcal{A}, h)$ induces the functor $\mathcal{A}$ and so by Theorem 2.3.2 $\mathcal{A}$ is a modular operad. $\square$

Corollary 2.4.5. Let $\mathcal{A}$ and $\mathcal{A}'$ be modular operads. A morphism of stable Σ-modules $\eta : \mathcal{A} \to \mathcal{A}'$ is a morphism of modular operads if, and only if, η commutes with the contraction maps and operadic composition.

Proof. This is just a restatement of Corollary 2.3.3 □

Chapter 3

The stable graphical category $\mathcal{U}_{st}$

In this section we define the stable graphical category $\mathcal{U}_{st}$ and sketch the relation between presheaves on this category and modular operads.

3.1 Feynman and stable Graphs

We shall use a combinatorial definition of a graph, known as a Feynman graph, which were introduced in [JK11]. For the rest of this section, fix some infinite set S.

Definition 3.1.1. A *Feynman graph* G is a functor from the category

$$i \circlearrowright A \xleftarrow{s} D \xrightarrow{t} V$$

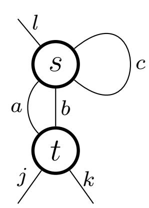
to the category of finite subsets of S and functions between them such that $G(i)$ is a fixed point free involution and $G(s)$ is a monomorphism.

The *vertices* of G are the elements of $G(V)$ and the *arcs* are the elements of $G(A)$. From now on these will just be denoted by V and A, and $G(D)$ by D. We can then form a category $\mathcal{G}$ of graphs, with the morphisms being natural transformations.

Remark. The requirement that A, D and V are subsets of S is a set-theoretic requirement to ensure that $\mathcal{G}$ is a small category

The set D in the above definition can be thought of as the set of arcs in G that are attached to a vertex. The partial function t then sends this arc to the vertex it is attached to (if it exists). The involution i is the function that swaps the orientation of an arc a. Thus we define an *edge* to be a pair $\{a, i(a)\}$. We often write $a^\dagger$ for $i(a)$. It is important to note that this definition does not require an arc to be attached to a vertex.

Example 3.1.2. Here is an example of a topological graph



As a Feynman graph, it is defined by $V = \{s, t\}$, $D = \{a, i(a), b, i(b), c, i(c), i(j), i(k), i(l)\}$ and $A = D \cup \{j, k, l\}$.

Given a Feynman graph G we can construct a topological graph $|G|$. Recall from Definition 1.3.1 that a topological graph is a pair (X, V) where X is a one-dimensional CW-complex and $V \subseteq X$ a finite subset of X such that $X \setminus V$ is a 1-manifold with finitely many connected components. This construction will be made explicit in Lemma 3.1.3.

Similarly, given a topological graph (X, V) we can construct a Feynman graph as follows: Take V to be as given and let $A = \pi_0(X \setminus V) \coprod \pi_0(X \setminus V)$. For each element $a \in \pi_0(X \setminus V)$ choose a particular orientation and label it $(a, 0)$. Let $(a, 1)$ be a with the opposite orientation, and define $i(a, j) = (a, j + 1)$ where addition takes place modulo 2. Define a partial function $t : A \to V$ by $(a, i) \mapsto \lim_{x \to 1} f(x)$, where $f : (0, 1) \to a$ is a parametrization of the 1-manifold a respecting the chosen orientation. Let D be the domain of t and s the inclusion of D in A. We say that a topological graph (X, V) is a realization of the Feynman graph G if G is induced by (X, V) in this way.

Note however that the Feynman graph given by $A = \{a, a^\dagger\}$ and $D = V = \emptyset$ does not have a unique realization. Both $(S^1, \emptyset)$ and $((0,1), \emptyset)$ induce this Feynman graph. However, this is the only time this can occur, as outlined in the following lemma:

Lemma 3.1.3. Suppose that G is a Feynman graph such that for every pair of arcs $\{a, a^\dagger\}$ at least one of $t(a)$ or $t(a^\dagger)$ exists. Then there is a unique (up to isomorphism) topological graph $|G|$ that induces the Feynman graph G

Proof. Let $|V|$ be any collection of $\#V$ points in $\mathbb{R}^3$ and let $f : V \to |V|$ be any bijection. Given an arc $a \in A$, if $t(a)$ exists attach a half open line segment to $f(t(v))$. If $t(a)$ does not exist, do nothing. Finally, if both $t(a)$ and $t(a^\dagger)$ exist, form the quotient space by identifying the two corresponding line segments. Call the resulting space $|G|$.

Note that the condition that at least one of $t(a)$ or $t(a^\dagger)$ exists guarantees that every arc in G is induced by one in X. This is because the edge corresponding to either one of these will have two orientations, one corresponding to a and the other to $a^\dagger$. Considering that gluing instructions uniquely determine a CW-complex, it is clear that this construction is unique and induces V and t correctly. $\square$

To finish off, we give several important properties and objects related to graphs, starting with connectedness.

Definition 3.1.4. A Feynman graph G is connected if the functor $\mathrm{Hom}_{\mathcal{G}}(G, -) : \mathcal{G} \to \mathbf{Set}$ preserves coproducts.

This definition mirrors the statement that a topological space X is connected if and only if $\mathrm{Hom}_{\mathbf{Top}}(X, -)$ preserves coproducts.

Definition 3.1.5.

- The *neighbourhood* of a vertex $v \in V$ is $nb(v) := t^{-1}(v)$.

- The *boundary* of a graph G is $\partial(G) := A \setminus D = A \setminus \coprod_{v \in V} nb(v)$.
- The *valence* of a vertex v is $|v| = |nb(v)| = n(v)$. Similarly, let $n(G) = |\partial(G)|$.

Example 3.1.6. Recall the graph G in Example 3.1.2. The boundary of G is $\partial(G) = \{i, j, k\}$.

Definition 3.1.7. The corolla with n legs, denoted C_n is the graph defined by $V = \{v\}$, $A = \{a_1, a_1^\dagger, ..., a_n, a_n^\dagger\}$ and $nb(v) = \{a_1, ..., a_n\}$.

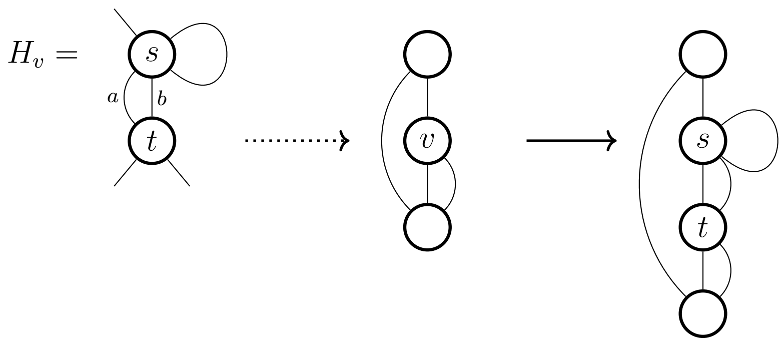
Given a vertex v in a graph G, the corolla C_v has v as its only vertex and $A = nb(v) \coprod i(nb(v))$. For $a \in nb(v)$, we define $t(a) = v$ and leave $t(a)$ undefined for $a \notin nb(v)$. The involution i is defined in the evident way.

Example 3.1.8.

The left graph here is $*_3$, and the graph on the right is $*_s$ for the vertex s in the graph from Example 3.1.2.

Finally, to motivate the eventual definition of a graphical map, we need to introduce the notion of graph substitution. Suppose that G is a stable graph and $\{H_v\}_{v \in V}$ is a collection of stable graphs satisfying $\partial(H_v) \cong i(nb(v))$ for all $v \in V$. The substituted graph $G\{H_v\}$ is given by replacing each vertex v with H_v, as in the following example:

Example 3.1.9.



Here the rightmost graph is the result of substituting H_v into the vertex v.

3.2 Graphical maps

Here we will define graphical maps. These morphisms model the idea of substituting the collection $\{H_v\}_{v \in V}$ into the graph G. For our purposes, H_v will represent surfaces and this substitution represents the process of gluing these surfaces together using the graph G as gluing instructions. For now, we need to formalise the idea of a subgraph, which leads us to embeddings.

Definition 3.2.1. An *étale map* is a morphism of graphs $f : G' \to G$ such that the rightmost square in the following diagram is a pullback.

$$\begin{array}{ccccc} i' \circlearrowright A' & \longleftarrow & D' & \xrightarrow{t'} & V' \\ \downarrow & & \downarrow f_D & & \downarrow f_V \\ i \circlearrowright A & \xleftarrow[t]{} & D & \longrightarrow & V \end{array}$$

The étale condition says that if $v \in f(V')$ and $a \in t^{-1}(v)$, there must exist an $a' \in D'$ such that $f(a') = a$. In this sense, it can be seen that an étale map is locally valence preserving; hence the name. Now we must define what it means to embed a graph G' into another graph G.

Definition 3.2.2. An étale map $f : G' \to G$ is an *embedding* if f_V is a monomorphism.

Thus an embedding is a local isomorphism that is injective on vertices. When G is connected we write $\widetilde{Emb}(G)$ for the set of all embeddings $f : G' \to G$, with G' a connected graph. In a sense this can be seen as the set of subgraphs of G'. Equivalently, one can think of $\widetilde{Emb}(G)$ as giving the possible ways of building G via graph substitution. As noted above, this is in fact how we will use this set in future.

Lemma 3.2.3. Suppose that the following diagram is a pullback in a category $\mathcal{C}$:

$$\begin{array}{ccc} P & \xrightarrow{p_2} & A \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{f} & C \end{array}$$

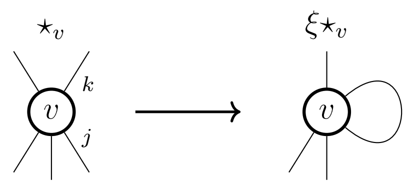
Then p_1 is a monomorphism if g is.

Proof. Suppose that $h_1, h_2 : D \to P$ are morphisms with $p_1 h_1 = p_1 h_2$. We then have $g p_2 h_1 = f p_1 h_1 = f p_1 h_2 = g p_2 h_2$. Hence $p_2 h_1 = p_2 h_2$ as g is a monomorphism. Therefore $h_1 = h_2$ by the universal property of the pullback. $\square$

Corollary 3.2.4. Let $f : G' \to G$ be an embedding. Then $f_D : D' \to D$ is injective.

Hence embeddings are also injective on internal edges. They need not be injective on arcs, however.

Example 3.2.5.



Note that here k is identified with $j^\dagger$ and $k^\dagger$ with j. However, since $j^\dagger \notin \partial(G)$ no two legs are identified with each other.

Corollary 3.2.6. Let $f : G' \to G$ be an embedding with G' and G both connected. Then the composite $\partial(G) \hookrightarrow A' \xrightarrow{f} A$ is injective.

Proof. First, if G' is the exceptional edge, we have $\partial(G') = \{a, a^\dagger\}$ and $f(a^\dagger) = f(a)^\dagger \neq f(a)$ and so f is injective on $\partial(G')$. The same argument holds for if G' is the exceptional loop, so assume that G' is neither. Then, as G' is connected, we must have $i\partial(G') \subseteq s(D')$. Hence every map in the composition

$$\partial(G') \to s(D') \xrightarrow{s'^{-1}} D' \xrightarrow{f_D} D \xrightarrow{s} s(D) \to \partial(G)$$

is a monomorphism, and so the composed function is injective. This composition is exactly the action of f on $\partial(G')$, as required. $\square$

From now on we write $\partial(f)$ for the image of $\partial(G')$ under f.

Lemma 3.2.7. Let $f : G' \to G$ be an embedding of connected graphs, where $\partial(G') = \emptyset$. Then f is an isomorphism.

Proof. By connectivity of G to show that f_V is surjective it suffices to show that if $v \in V$ is adjacent to $f(v')$ for some $v' \in V'$ then $v \in f(V')$. Let $a \in nb(f(v'))$ be such that $a^\dagger \in nb(v)$. Then $a \in t^{-1}(f(v'))$ and so there exists a $d' \in D'$ such that $f(d') = a$ and $t'(d') = v$ by the étale condition on f.

Then $a^\dagger = (f(d'))^\dagger = f(d'^\dagger)$. Since $A' = D'$, $\overline{v} = t'(d'^\dagger)$ is well defined. We then have $f(\overline{v}) = f(t'(d'^\dagger)) = t(f(d'^\dagger)) = t(a^\dagger) = v$. Thus f_V is bijective. Thus so is f_D by the étale condition. Finally, as $A' = D'$ and $f_A = f_D$ is a bijection, we have that $A = D$ by a simple counting argument. $\square$

Theorem 3.2.8. *If $f : G' \to G$ and $h : G'' \to G$ are in $\widetilde{Emb}(G)$ with neither G' nor G'' the exceptional edge and $\partial(f) = \partial(h)$, there is a unique isomorphism $z : G' \to G''$ such that $f = hz$.*

Proof. First we show existence. If $\partial(G')$ or $\partial(G'')$ are empty, f and h are both isomorphisms by Lemma 3.2.7 and so setting $z = h^{-1}f$ works. So for now we assume that both G' and G'' have non-empty boundary. As h is a monomorphisms on $\partial(G'')$ and $\partial(f) = \partial(h)$, we can set $z(a) = h^{-1}(f(a))$

for all $a \in \partial(G')$. Similarly, set $z(a^\dagger) = z(a)^\dagger$ for all $a \in \partial(G')$. Since G'' is not the exceptional edge and $z(a) \in \partial(G'')$, $z(a^\dagger) \in D''$ and so we can set $z(v) = t''(z(a))$, where $a^\dagger \in \partial(G')$ and $v = t'(a)$.

Now suppose that we have defined $z(v)$ and $a \in nb(v)$. Then $f(v) = h(z(v)) \in h(V'')$ and similarly $f(a) \in nb(f(v))$. Hence by the étale condition on h there exists a unique $a' \in D''$ such that $h(a') = f(a)$. Thus we can set $z(a) = a'$ and $z(a^\dagger) = z(a)^\dagger$. Hence if we have defined z at a vertex $v \in V'$ we can also define it on $nb(v) \cup nb(v)^\dagger$. Similarly, if we have defined $z(a)$ for $a \in D'$ we can define $z(t'(a)) = t''(z(a))$ as $z(a) \in D''$. By connectivity of G' this process defines a map $z : G' \to G''$ and it is clear that $f = hz$.

Now, to see uniqueness, simply note that as h_V and f_V are both monomorphisms there is at most one function z_V such that $f_V = h_V z_V$. A similar argument holds for z_D and $z_A|_{\partial(G')}$, and so z_A is also unique as $A' = D' \cup \partial(G')$. □

Remark. The result also holds if both G' and G'' are the exceptional edge. In this case z is either the identity or i.

In light of Theorem 3.2.8, we can see that $\widetilde{Emb}(G)$ contains a large amount of unnecessary information. We therefore quotient out by the equivalence relation $\sim$ given by $f \sim h$ if there exists an isomorphism z such that $f = hz$ to form the set $Emb(G)$.

We are now nearly ready to give the definition of a graphical map. Before we do, we give a quick preliminary definition

Definition 3.2.9. Let $\mathbb{N}V$ denote the free commutative monoid on V. Given any graph G we can define a function $\mathcal{D} : Emb(G) \to \mathbb{N}V$ by $f \mapsto \Sigma_{v \in V'} f(v)$.

Note that as an embedding is an injection on vertices we always have $\mathcal{D}(f) \leq \Sigma_{v \in V} v$. We can now define a graphical map.

Definition 3.2.10. A *graphical map* $\varphi : G \to G'$ is given by a pair of functions $\varphi_0 : A \to A'$ and $\varphi_1 : V \to Emb(G')$ satisiying the following axioms:

- $\Sigma_{v \in V} \mathcal{D}(\varphi_1(v)) \leq \Sigma_{w \in V'} w$
- For each $v \in V$ there exists an isomorphism $nb(v) \to \partial\varphi_1(v)$ which makes the following diagram commute:

$$\begin{array}{ccc} nb(v) & \xrightarrow{i} & A \\ \downarrow & & \downarrow{\scriptstyle \varphi_0} \\ \partial(\varphi_1(v)) & \xrightarrow{i} & A' \end{array}$$

- If the boundary of G is empty then there is an edge $v \in V$ such that $\varphi_1(v)$ is not given by embedding an edge

This definition can be interpreted as saying that a subgraph of G' can be obtained from G by graph substitution; specifically by substituting the graph $\varphi_1(v)$ into the vertex v. To see this, note that the first condition guarantees that the images of the $\varphi_1(v)$ are disjoint, just as the result of substitution would be. Similarly, the second condition guarantees that it is possible to substitute $\varphi_1(v)$ into v, as the output edges are in a one-to-one correspondence. The last condition is to avoid collapse.

Definition 3.2.11. The *graphical category* $\mathcal{U}$ is the category with objects all Feynman graphs, excluding the nodeless loop, and morphisms the graphical maps. See [HRY18] Theorem 2.27 for a proof that this is in fact a category

Lemma 3.2.12. If $\varphi : G \to G'$ is a graphical map write $\varphi_1(v) = H_v \to G'$ for the embeddings. Then there is an embedding $k : G\{H_v\}_{v \in V} \to G'$.

Proof. Write

$$i \circlearrowright A_v \xleftarrow{s} D_v \xrightarrow{t} V_v$$

for H_v. Then $G\{H_v\}_{v \in V}$ has $V = \coprod_{v \in V} V_v$, $D = \coprod_{v \in V} D_v$ and $A = \coprod_{v \in V} A_v / \sim$ where $a \sim a'$ if $a \in nb(t(a'))$. On V and D we can define k

by $\coprod_{v \in V} \varphi_1(v)$ and we can extend this to A by using φ_0.

The first condition in the definition of graphical maps shows that $V \to V'$ is injective. Finally, k is étale because each $\varphi_1(v)$ is. $\square$

Definition 3.2.13. The *image* of a graphical map $\varphi : G \to G'$ is the embedding k defined in Lemma 3.2.12 and is denoted by $im(\varphi)$.

3.3 The stable graphical category

In this section we define stable graphs and modify our graphical category to make it better suited for studying surfaces.

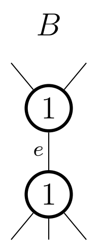
Recall the original motivation. Each vertex of a graph G should represent a surface S, where the valence of v is the number of boundary components. For this to make sense, every vertex should have some genus data on top of the valence. This leads to labelled and stable graphs.

Definition 3.3.1. An *ordering* of a graph G is a bijection $\rho : \mathbb{Z}/n\mathbb{Z} \to \partial(G)$

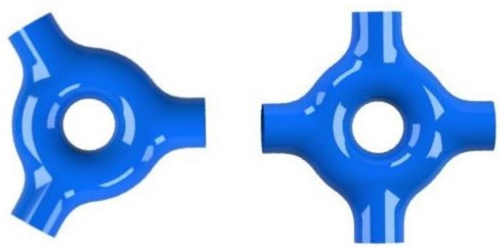
Definition 3.3.2. A *labelled graph* G is a Feynman graph with an ordering ρ and a function $g : V \to \mathbb{Z}_{\geq 0}$. A labelled graph G is called *stable* if for every vertex $v \in V$, $2g(v) - 2 + n(v) > 0$, where $n(v) = |t^{-1}(v)|$. The genus $g(f)$ of an embedding $f : G' \to G$ is the genus of the domain.

As noted previously, the function g can be thought of as giving the genus of the vertex v. To expand on that remark, for each $v \in V$ let F_v be the surface with genus $g(v)$ and $n(v)$ boundary components. The graph G can then be thought of as instructions on how to glue these surfaces together. Specifically, F_v and F_w will have a pair of boundary components glued together for every edge in $nb(v)^\dagger \cap nb(w)$.

Example 3.3.3. The graph



represents the operation of gluing the two surfaces



along their boundaries.

Definition 3.3.4. The *genus* of a stable graph G is

$$g(G) = \sum_{v \in V} g(v) + \beta_1(|G|),$$

where $\beta_1(X) = \dim_{\mathbb{Q}}(H_1(X; \mathbb{Q}))$ is the first betti number of X.

Definition 3.3.5. A *stable graphical map* $\varphi : G \to G'$ between stable graphs G and G' is a graphical map that preserves the ordering on legs and satisfies $g(v) = g(\phi_1(v))$ for all $v \in V$.

A stable graphical map $\varphi : G \to G'$ is called *active* if it induces a bijection $\partial(G) \to \partial(G')$.

Lemma 3.3.6. A stable graphical map $\varphi : G \to G'$ is active if, and only if, $im(\varphi)$ is an isomorphism.

Proof. Note that the reverse implication is immediate from the definitions. For the forwards direction, we split into two cases: When G is the exceptional edge and when it is not.

For the first case, let $\{a, i(a)\}$ be the set of arcs in G. Suppose for a contradiction that φ is not an isomorphism. Then there is a vertex $v \in V'$ with an arc $b \in nb(v)$ such that $i(b) \in \partial(G')$. By the active assumption, we can also assume that $i(b) = \varphi_0(a)$. Then $b = i\varphi_0(a) = \varphi_0(ia)$ so that $b \notin nb(v)$, a contradiction.

For the general case, apply Theorem 3.2.8 to $im(\varphi)$. □

Corollary 3.3.7. If $\varphi : G \to G'$ is an active stable graphical map $g(G) = g(G')$

Proof. Simply note that by Lemma 3.3.6 $g(G') = g(im(\varphi)) = g(G)$. □

Definition 3.3.8. The *stable graphical category* $\mathcal{U}_{st}$ is the category with objects all stable graphs and morphisms the active stable maps between them.

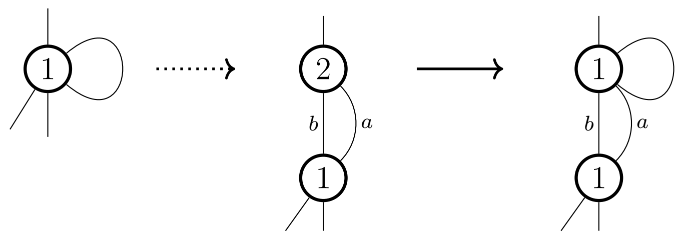
3.4 Special graphical maps and factorization

Here we define inner coface maps and state (without proof) the factorization theorem for morphisms in $\mathcal{U}_{st}$.

Definition 3.4.1. Let G be a stable graph and let $v \in V$. An *inner coface map* is a graphical map given by substituting a graph H_v with one internal edge into v and corollas into all other vertices.

There are two possible types of inner coface maps. The first is when H_v is a contracted corolla. In this case, H_v has one vertex of genus $g(v) - 1$, $nb(v)$ legs and one loop. Substituting H_v into v can be viewed as 'pulling out' the genus in the vertex v. As we will be considering contravariant functors, this can conversely be viewed as contracting the arcs in $nb(v)$ corresponding to the loop and pulling the genus data of the loop into the vertex v.

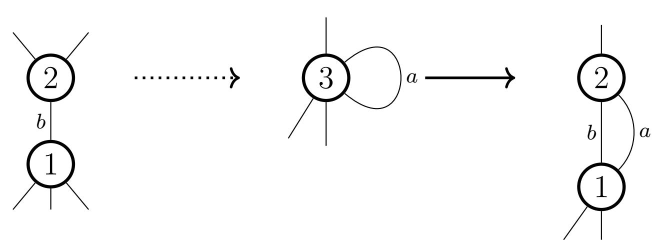
Example 3.4.2.



The above shows an inner coface map given by substituting a contracted corolla into the vertex with genus 2 in the middle graph. Note that this lowers the genus of this vertex and introduces a loop.

The second case is when H_v is a barbell; a graph given by gluing two corollas together along a single edge. In this case, H_v has two vertices u and w satisfying $g(v) = g(u) + g(w)$. Similarly to before, this can be viewed as seperating the genus in v into two parts. Conversely, it can be viewed as contracting the edge connecting u to w and collecting their genus data into one vertex.

Example 3.4.3.



This shows an inner coface map given by substituting a barbell. Note that this splits up the genus in the vertex.

We have the following factorisation theorem:

Theorem 3.4.4. *Every morphism in $\mathcal{U}_{st}$ factors as a composition of inner coface maps followed by an isomorphism.*

Proof. This is just the factorisation theorem in [HRY18]. The outer coface maps and codegeneracies defined in [HRY18] are not necessary as they are not morphisms in $\mathcal{U}_{st}$. □

Geometrically, this theorem can be interpreted as saying that every graph of genus g with n boundary components can be built by substituting barbells and contracted corollas into the corolla $*_{g,n}$.

3.5 $\mathcal{U}_{st}$ and modular operads

Here we will sketch the relationship between modular operads in a symmetric monoidal category $\mathcal{C}$ and functors $\mathcal{U}_{st}^{op} \to \mathcal{C}$. Specifically, we will show that $\mathcal{U}_{st}$ is opposite to Γ and use this to demonstrate that inner coface maps given by substituting barbells can be interpreted as operadic composition and those given by substitution of a contracted star can be interpreted as contraction maps.

Theorem 3.5.1. *$\mathcal{U}_{st}^{op}$ is equivalent to Γ.*

Proof. Define a functor $F : \mathcal{U}_{st}^{op} \to \Gamma$ by sending an object (G, ρ) to the graph $F(G)$ consisting of the following data:

- $\mathrm{Flag}(G) = A$.
- The involution σ is given by the involution i.
- The partition λ defined by the equivalence relation $a \sim a'$ if $a, a' \in nb(v)$ for some $v \in V$.
- The ordering on legs is given directly by ρ.

Next, let $\varphi : G \to G'$ be a graphical map. As φ is active, G' is isomorphic to $G\{H_v\}$ for some collection $\{H_v\}_{v \in V}$ by Lemma 3.3.6. Define $F(\varphi) : F(G') \to F(G)$ to be the morphism of graphs given by contracting the subgraphs H_v. This is clearly functorial, faithful and essentially surjective.

To see that it is full, first recall that every morphism in Γ is a composition of contractions of edges. Let $f : G \to G/I$ be such a map. If I formed a loop at a vertex v, then f is exactly the image of substituting a contracted corolla into v. Similarly, if I was not an edge, then f is the image of substituting a barbell into v. Hence F is full.

□

Now, recall that modular operads are defined on corollas $*_{g,n}$ and then extended to general graphs using the tensor product in $\mathcal{C}$. General presheafs on $\mathcal{U}_{st}$ have no such restriction, making the prospect of interpreting a general contravariant functor as a modular operad hopeless. Instead, we retrict attention to functors which are generated by their values on corollas:

Definition 3.5.2. Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. A contravariant functor $F : \mathcal{U}_{st} \to \mathcal{C}$ satisfies the strict Segal condition if for all $G \in \mathcal{U}_{st}$ we have $F(G) \cong \otimes_{v \in V} F(*_v)$.

Let $\mathcal{MC}$ be the category with objects presheafs $F : \mathcal{U}_{st}^{op} \to \mathcal{C}$ on $\mathcal{U}_{st}$ satisfying the strict Segal condition. As mentioned above, $\mathcal{MC}$ can be viewed as an extension of $\mathbf{Mod}\mathcal{C}$, as shown by the following theorem.

Theorem 3.5.3. *Let $(\mathcal{C}, \otimes, I)$ be a distributive symmetric monoidal category with all finite colimits. Then there is a fully faithful functor $F : \boldsymbol{Mod}\mathcal{C} \to \mathcal{MC}$.*

Proof. Let $\mathcal{A}$ be a modular operad in $\mathcal{C}$ and let $F : \mathcal{U}_{st}^{op} \to \Gamma$ be the categorical equivalence defined in Theorem 3.5.1. Define $F_{\mathcal{A}}$ to be the composition $\mathcal{U}_{st}^{op} \overset{F}{\to} \Gamma \overset{\mathcal{A}((-))}{\to} \mathcal{C}$.

This is contravariant and satisfies the strict Segal condition. By Corollary 2.3.3 a morphism of modular operads $\eta : \mathcal{A} \to \mathcal{A}'$ induces a natural transformation $\mathcal{A}((-)) \to \mathcal{A}'((-))$ and hence induces a natural transformation $F_{\mathcal{A}} \to F_{\mathcal{A}'}$. Thus $\mathcal{A} \mapsto F_{\mathcal{A}}$ is a functor $\mathbf{Mod}\mathcal{C} \to \mathcal{MC}$.

Fullness follows from fullness of F and Corollary 2.3.3. To see faithfulness note that the natural transformation η is uniquely defined by its value on $*_{g,n}$ for all $g, n \geq 0$. This combined with faithfulness of F shows that $\mathcal{A} \mapsto F_{\mathcal{A}}$ is faithful.

□

Remark. The main obstruction in having this functor be an equivalence is the lack of an action of Σ_n on $F(*_{g,n})$. If we included this information in the definition of a functor satisfying the strict Segal condition (along with the necessary equivariance requirements) the functor F would in fact be an equivalence of categories.

Another approach is to remove the requirement that morphisms in $\mathcal{U}_{st}$ preserve the order. In this case we allow automorphisms that just permute the ordering on legs. We can use this to introduce a Σ_n action on $F(*_{g,n})$ which will also make the above functor into an equivalence of categories. The problem with this approach is that Theorem 3.5.1 will no longer hold as the resulting hom sets will be a direct product of the original hom sets with Σ_n.

Chapter 4

Homotopy theory and profinite completions

In this section we review some abstract homotopy theory. We give the segal conditions for modular operads and discuss when the profinite completion of a modular operad results in an ∞-modular operad.

4.1 The segal core of a graph

In this section we define the Segal core $\mathbf{Sc}[G]$ of a graph G. Let $\mathcal{U}_{st}[G]$ be the representable presheaf $\mathrm{Hom}_{\mathcal{U}_{st}}(-, G)$.

Definition 4.1.1. Fix a graph $G \in \mathcal{U}_{st}$. $\mathcal{C}^G$ is the category with objects $Ed(G) \coprod Vt(G)$ and all non-identity arrows $a : e \to v$ given by an arc a with underlying edge e such that $t(a) = v$.

Next, note that for any corolla $*_n$ and an arc a in $*_n$ with underlying edge e there is a natural transformation $a : \mathcal{U}_{st}[e] \to \mathcal{U}_{st}[*_n]$ given by postcomposing a morphism $G \to e$ with the embedding $e \hookrightarrow *_n$.

This allows us to define a functor $F^G : \mathcal{C}^G \to \mathbf{Set}^{\mathcal{U}_{st}^{op}}$ by sending an edge e to $\mathcal{U}_{st}[e]$ and the vertex v to $\mathcal{U}_{st}[*_v]$. The morphism $a : e \to v$ is then sent to the natural transformation $\mathcal{U}_{st}[e] \to \mathcal{U}_{st}[*_v]$ defined above.

Definition 4.1.2. The *Segal core* of G is the $\mathcal{U}_{st}$-presheaf

$$\mathbf{Sc}[G] = \underrightarrow{\operatorname{colim}}_{\mathcal{C}^G} F^G$$

Remark. Embedding vertices into the graph G defines a map $\mathbf{Sc}[G] \to \mathcal{U}_{st}[G]$ that identifies the Segal core with the union of $\mathcal{U}_{st}[*_v]$ over all vertices v of G. If we restrict to the case of a $\mathcal{U}_{st}$-presheaf in simplicial sets, this (along with the Yoneda lemma) allows us to identify the mapping space $map(\mathbf{Sc}[G], X)$ with the product of X_{*_v} over all vertices v in G. This is used in Definition 4.4.1

4.2 Profinite completion

Profinite completion is a fundamental operation in the study of étale fundamental groups and so is naturally important in the study of the Teichmüller tower. In this section, we recall the definition of profinite completion of a group and of a space.

First, recall that a *pro-object* in a category $\mathcal{C}$ is a filtered limit of objects in $\mathcal{C}$. A *profinite group* is then a pro-object in the category of finite groups **Fin-Group**. That is, a profinite group is a filtered system of finite groups, though we often consider the limit of this diagram. We usually consider these finite groups as having the discrete topology which then gives a topology on their limit G. Similarly, a profinite topological space is a space X homeomorphic to a limit of finite, discrete spaces. We then have the following classical result:

Theorem 4.2.1. *A topological group (resp. space) G is profinite if and only if it is compact, hausdorff and totally disconnected.*

Given an arbitrary group G there is a canonical profinite group $\widehat{G}$ with a morphism $G \to \widehat{G}$.

Definition 4.2.2. Let $\mathcal{N}$ be the set of finite index normal subgroups of G. The *profinite completion* of G is defined by the formula

$$\widehat{G} := \varprojlim_{N \in \mathcal{N}} G/N \cong \{(x_N)_{N\in\mathcal{N}} \in \prod_{N\in\mathcal{N}} | X_N = X_{N'} \text{ (mod N) if } N' \subseteq N\}$$

where the limit is taken in the category of topological groups.

There is a morphism $G \to \widehat{G}$ given by $x \mapsto ([x]_N)_{N\in\mathcal{N}}$ where $[x]_N$ is the equivalence class of x modulo N. Clearly this map is injective if and only if $\bigcap_{N\in\mathcal{N}} N = 1$.

The profinite completion $\widehat{G}$ can also be characterised by a universal property:

Theorem 4.2.3. *Let G be a discrete topological group. If H is a profinite group and $\varphi : G \to H$ is a continuous group morphism, there is a unique continuous morphism $\widetilde{\varphi} : \widehat{G} \to H$ making the following diagram commute:*

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \downarrow & \overset{\widetilde{\varphi}}{\nearrow} & \\ \widehat{G} & & \end{array}$$

Proof. Let $H = \varinjlim_{i\in I} F_i$, where all F_i are finite groups. Consider the composition $G \to H \to F_i$. This factors as a continuous map $G/N_i \to F_i$, where N_i is the kernel of $G \to F_i$. Noting that G/N_i is necessarily finite, by precomposing with projection we get a continuous map $\widehat{G} \to F_i$ for all i.

Now, suppose that $i \to j$ is a morphism in I. The commuting diagram

$$\begin{array}{ccccc} G & \longrightarrow & H & \longrightarrow & F_i \\ & & & \searrow & \downarrow \\ & & & & F_j \end{array}$$

Tells us that $N_j \supseteq N_i$. Hence we have a commuting diagram

$$\begin{array}{ccccc}
\widehat{G} & \longrightarrow & G/N_i & \longrightarrow & F_i \\
 & \searrow & \downarrow & & \downarrow \\
 & & G/N_j & \longrightarrow & F_j
\end{array}$$

and so we find that these maps make $\widehat{G}$ into a cone over F_i. Thus there is a unique continuous morphism $\widetilde{\varphi} : \widehat{G} \to H$ such that

$$\begin{array}{ccccc}
\widehat{G} & & & & \\
 & \overset{\widetilde{\varphi}}{\searrow} & & & \\
 & & H & \longrightarrow & F_i \\
 & & \downarrow & \swarrow & \\
 & & F_j & &
\end{array} \qquad (\widehat{G} \to F_i,\ \widehat{G} \to F_j) \tag{1}$$

commutes. Furthermore,

$$\begin{array}{ccc}
G & \longrightarrow & H \\
\downarrow & \nearrow & \\
\widehat{G} & &
\end{array} \tag{2}$$

commutes because it commutes on all factors of H. Finally, if a map $\widehat{G} \to H$ makes (2) commute it necessarily makes (1) commute and so by uniqueness must be equal to $\widetilde{\varphi}$. □

There is a similar concept of profinite sets and simplicial sets. As for groups, a profinite set is a pro-object in the category of finite sets. The category of profinite sets is denoted by $\widehat{\mathbf{Set}}$. Given a set X, we can form a profinite set $\widehat{X}$ similarly for groups. Specifically, $\widehat{X}$ is the limit of $X/\sim$ as the equivalence relation $\sim$ varies over all equivalence relations such that $X/\sim$ is finite. This gives a functor $\mathbf{Set} \to \widehat{\mathbf{Set}}$.

Similarly a profinite simplicial set is a pro-object in the category of simplicial sets. The category of profinite simplicial sets is denoted by $\widehat{\mathbf{sSet}}$. The profinite completion $\widehat{X}$ of a simplicial set $X : \Delta^{op} \to \mathbf{Set}$ is just the composition $\Delta^{op} \xrightarrow{X} \mathbf{Set} \to \widehat{\mathbf{Set}}$.

4.3 Good groups

In this section we define *goodness* for groups. The importance of the concept in this thesis is due to Theorem 4.3.2 which says that products of spaces with good homotopy groups interacts well with profinite completion.

Definition 4.3.1. A group G is called *good* if given any finite G-module M the map $G \to \widehat{G}$ induces isomorphisms $H^q(\widehat{G}, M) \to H^q(G, M)$ for all $q \geq 0$.

The following theorem gives the primary motivation for the use of good groups in this thesis.

Theorem 4.3.2. *Let X and Y be spaces such that $\pi_i(X)$ and $\pi_i(Y)$ are good for all $i \geq 1$, then the map $\widehat{X \times Y} \to \widehat{X} \times \widehat{Y}$ is a homotopy equivalence.*

Proof. See proposition 3.9 in [BdBHR18] □

Now we present some basic results about good groups, which can be found in [GJ-ZZ08].

Lemma 4.3.3. Let G_1 and G_2 be good groups. Then $G_1 \times G_2$ is good.

Proof. See [GJ-ZZ08] Proposition 3.4 □

Theorem 4.3.4. *Let H be a group. Suppose there exists a short exact sequence*

$$1 \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow 1$$

such that G and N are good, N is finitely generated and $H^q(N, M)$ is finite for all finite H-modules M. Then H is good.

Proof. See [GJ-ZZ08] Lemma 3.3 □

Lemma 4.3.5. Let G be a group such that $K(G, 1)$ has finitely many cells in every dimension. Then the cohomology groups $H^q(G, M)$ are finite for all finite G-modules M.

Proof. Choose a particular model for $K(G,1)$ and let EG be the corresponding universal cover. Let $C_*(EG)$ be the augmented cellular chain complex of EG. $C_*(EG)$ then has a free action of G induced by the action of G on EG. As EG is contractible, this makes $C_*(EG)$ into an exact sequence of free $\mathbb{Z}[G]$-modules, ending in the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$.

Thus the cohomology groups $H^q(G,M)$ is the cohomology of the chain complex $Hom_{\mathbb{Z}[G]}(C_*(EG),M)$. Note that $C_n(EG)$ is finitely generated as a $\mathbb{Z}[G]$-module; its rank is the number of n-cells in $K(G,1)$. Hence $Hom_{\mathbb{Z}[G]}(C_n(EG),M)$ is finite if M is finite. Thus $H^q(G,M)$ must be finite as it is a quotient of a subgroup of a finite group. □

Remark. In particular, Lemma 4.3.5 shows that if $K(N,1)$ is finite we only need to check that N is good to apply Theorem 4.3.4. In particular, we will use this for finitely generated free groups, whose classifying spaces are wedges of circles, and for surface groups whose classifying spaces are the corresponding closed surfaces.

4.4 ∞-modular operads

Here we define ∞-modular operads and give some situations when the profinite completion of a strict modular operad in simplicial sets gives an ∞-modular operad.

Definition 4.4.1. A Segal ∞-*modular operad* in **sSet** is a presheaf $P : \mathcal{U}_{st}^{op} \to$ **sSet** which satisfies the following three axioms:

- X_I is contractible, where I is the exceptional edge
- X_G is fibrant for all graphs G. That is, X_G is a Kan complex.
- The map $X_G \to \prod_{v\in V} X_{*_v}$ is a weak equivalence of simplicial sets.

Remark. This is equivalent to saying that X is fibrant in a particular model category structure on the presheaf category $\mathbf{sSet}^{\mathcal{U}_{st}^{op}}$. See [HRY18] Theorem 4.10 for more details.

Theorem 4.4.2. *Let X be a $\mathcal{U}_{st}$-presheaf satisfying the strict Segal condition in the category of groups **Grp** and let B : **Grp** $\to$ **sSet** be the classifying space functor. Suppose that X_G is good for all graphs G and $X_I = e$ where e is the trivial group. Then the profinite completion of BX is an ∞-modular operad.*

Proof. It is a result in [Qui08] that if H is a good group, $B\widehat{H} \simeq \widehat{BH}$. Hence $\widehat{BX_G}$ is fibrant for all G as classifying spaces are fibrant.

Next, recall that the classifying space BH satisfies $\pi_1(BH) = H)$ and $\pi_i(BH) = 0$ for $i > 1$. Hence if H is good all homotopy groups of BH are good. Thus by Theorem 4.3.2 the map

$$\widehat{BX_G} \cong \widehat{\prod_{v \in V} BX_{*_v}} \to \prod_{v \in V} \widehat{BX_{*_v}}$$

is a weak equivalence, as required.

Finally, note that Be is the simplicial set given by $Be_0 = \{e\}$, and $B\{e\}_i = \{e \stackrel{id_e}{\to} ... \stackrel{id_e}{\to} e\}$ for $i > 0$. The profinite completion of Be is then just Be, which is contractible. $\square$

Chapter 5

The functor Γ

In this section we construct a functor $\Gamma : \mathcal{U}_{st}^{op} \to \mathbf{Grpd}$ using a modification of Tillmann's higher Genus surface operad.

5.1 The groupoids $\mathcal{S}_{g,n}$

In [Til00], Tillmann constructs a groupoid $\mathcal{S}_{g,n,1}$ and uses them to form an operad called the *higher genus surface operad*. The construction presented here is simplified as our definition of a cyclic operad does not include the unital axiom.

Definition 5.1.1. Let S be a surface with n boundary components. An *ordering* ρ of S is a bijection $\rho : \mathbb{Z}/n\mathbb{Z} \to \pi_0(\partial S)$.

Now, given two labelled surfaces $(S_{g,n}, \rho)$ and $(F_{h,m}, \rho')$ we can form a new labelled surface $(S_{g,n} \circ_{ij} F_{h,m}, \rho \circ_{ij} \rho')$ given by pasting the ith boundary component of $S_{g,n}$ to the jth boundary component of $F_{h,m}$. As elements of the mapping class group are isotopy classes of homeomorphisms that fix the boundary, by viewing the original surfaces as submanifolds, we get a map $\circ_{ij} : \Gamma_{g,n} \times \Gamma_{h,m} \to \Gamma_{g+h,m+n-2}$.

Note that there are several problems with the above discussion. For one, we clearly cannot consider all surfaces of genus g with n boundary components as then we would not have a set of objects. Furthermore, we cannot just arbitrarily choose one surface for every g and n as then gluing our choices of surface $S_{g,n}$ and $S_{h,n}$ may not produce the surface we chose for $S_{g+h,n+m-2}$. To fix this, we instead only fix small atomic surfaces. All other surfaces can be built up by gluing these together.

To make the above rigorous, fix a surface of genus 0 with three boundary components (referred to as a pair of pants) and call it $F_{0,3}$ and give it an ordering ρ_1. Similarly, fix a surface of genus 1 with 2 boundary components $F_{1,2}$ and a disk $F_{0,1}$ with orderings ρ_2 and ρ_3 respectively. Also, fix some ϵ and embeddings $\phi_i^1 : S^1 \times [0, \epsilon) \to F_{0,3}$ such that $\phi_i^1(S^1 \times \{0\})$ is the ith boundary component of $F_{0,3}$. Similarly set embeddings $\phi_i^2 : S^1 \times [0, \epsilon) \to F_{1,2}$ and $\phi_0^3 : S^1 \times [0, \epsilon) \to F_{0,1}$.

We can glue these surfaces together using the relations given by $\phi_i^m(\phi_j^n)^{-1}$. To be precise the result of gluing the ith boundary component of S to the jth boundary component of F is $S \coprod F/ \sim_\alpha$, where the equivalence relation $\sim_\alpha$ is defined by $x \sim_\alpha y$ if x is an element of the ith boundary component of S and $y = \phi_j^F \circ (\phi_i^S(x))^{-1}$.

We can also allow gluing the boundary components of the same surface together. In this case the result would be $S/ \sim_\beta$ where $\sim_\beta$ is given by $x \sim_\beta y$ if x is in the ith boundary component of S and $y = \phi_j^S \circ (\phi_i^S)^{-1}(x)$.

Definition 5.1.2. We define a collection of groupoids $\{\mathcal{S}\}_{g,n\geq 0}$ as follows: If $2g - 2 + n \leq 0$ then $\mathcal{S}_{g,n} = \emptyset$. Otherwise the groupoid $\mathcal{S}_{g,n}$ has objects (S, ρ) where S is a surface of genus g with n boundary components constructed by gluing atomic surfaces as in the above discussion and ρ is an ordering of the boundary components of S.

Morphisms in $\mathcal{S}_{g,n}$ are given by isotopy classes of homeomorphisms which fix the boundary components point-wise (with respect to the identification given by $\phi_j\phi_i^{-1}$) and preserves the orderings.

Note that the full subcategory with one object (S, σ) is a skeleton of $\mathcal{S}_{g,n}$ and its hom set is given by the mapping class group $\Gamma_{g,n}$. Hence the inclusion of this subcategory into $\mathcal{S}_{g,n}$ is an equivalence of categories. Then as the nerve functor is a Quillen adjunction (and since all categories are fibrant) the classifying space of $\mathcal{S}_{g,n}$ is weakly equivalent to the classifying space of $\Gamma_{g,n}$.

5.2 Operadic composition and contraction maps for $\mathcal{S}_{g,n}$

As described at the start of this section, there are natural maps $\circ_{ij} : \mathcal{S}_{g,n} \times \mathcal{S}_{h,m} \to \mathcal{S}_{g+h,n+m-2}$. These send a pair $((S, \rho), (F, \rho'))$ to the surface given by gluing the ith boundary component of S to the jth boundary component of F. The ordering is given by following ρ until i, then following ρ' and then following the rest of ρ. That is,

$$(\rho \circ_{ij} \rho')(k) = \begin{cases} \rho(k) & 0 < k < i \\ \rho'(k - i + j) & i \leq k < i + m \\ \rho(k - m) & i + m \leq k \leq m + n \end{cases}$$

Note that both S and F are subsurfaces of the resulting surface.

On morphisms, $\circ_{ij}$ sends a pair $([f], [g])$ to the isotopy class of the isomorphism that acts as f on the subsurface S and as g on the subsurface F. This is well defined because both f and g act as the identity on the boundary of S and F.

This is the operadic composition for $\mathcal{S}_{g,n}$. These turn $\mathcal{S}_n = \coprod_{g \geq 0} \mathcal{S}_{g,n+1}$ into a stable graded cyclic operad.

Lemma 5.2.1. $\{\mathcal{S}\}_{n \geq 0}$ is a stable graded cyclic operad.

Proof. Recall that Σ_n acts on an element $(S, \rho) \in \mathcal{S}_n = \coprod_{g \geq 0} \mathcal{S}_{g,n+1}$ by permuting the non-zero labels.

First, we show that $\{\mathcal{S}_n\}_{n\geq 0}$ is indeed an operad. To define the composition maps $\circ_i$, note that $\mathcal{S}_n \times \mathcal{S}_m \cong \coprod_{g,h\geq 0} \mathcal{S}_{g,n+1} \times \mathcal{S}_{h,m+1}$. Then consider $\circ_{0i} : \mathcal{S}_{g,n} \times \mathcal{S}_{h,m} \to \mathcal{S}_{g+h,n+m-2}$. Composing with the inclusion $\mathcal{S}_{g+h,n+m-2} \hookrightarrow S_{n+m-1}$ and summing over all g, h we get a map $\circ_i : \mathcal{S}_n \times \mathcal{S}_m \to \mathcal{S}_{n+m-1}$.

Now we need to verify that these maps make $\{\mathcal{S}_n\}_{n\geq 0}$ into a cyclic operad. Let $(S, \rho) \in \mathcal{S}_n$, $(F, \rho') \in \mathcal{S}_m$, $\sigma \in \Sigma_n$ and $\pi \in \Sigma_m$. Suppose we glue the $\sigma(i)$th boundary component of σS to the 0th boundary component of πF. That is, consider the surface $(\sigma S) \circ_{\sigma(i)} (\pi F)$

Now suppose that j is such that $1 \leq j < i$ and $\sigma(j) < \sigma(i)$. In this case, the jth boundary component of $(\sigma S) \circ_{\sigma(i)} (\pi F)$ is simply the $\sigma(j)$th boundary component of $S \circ_i F$. If instead $\sigma(j) > \sigma(i)$, the jth boundary component of $(\sigma S) \circ_{\sigma(i)} (\pi F)$ comes after the boundary components of πF and so must be the $(\sigma(j)+m-1)$th boundary component of $S \circ_i F$.

If we have that $i \leq j < i + m$ then the jth boundary component of $(\sigma S) \circ_{\sigma(i)} (\pi F)$ lies in πF and is given by going $j - i + 1$ places past $\sigma(i) - 1$. Thus it must be the $(\sigma(i) + \pi(j - i + 1) - 1)$th boundary component of $S \circ_i F$.

Next suppose that j is such that $i + m \leq j \leq n + m$ and $\sigma(j) < \sigma(i)$. The jth boundary component of $(\sigma S) \circ_{\sigma(i)} (\pi F)$ is then before the ith boundary component of σS; it is given by going $j - m + 1$ places past 0. Hence the jth boundary component of $(\sigma S) \circ_{\sigma(i)} (\pi F)$ is the $\sigma(j - m + 1)$th boundary component of $S \circ_i F$. If $\sigma(j) > \sigma(i)$, this boundary component is past πF, and so is shifted ahead $m - 1$ places. Hence the jth boundary component of $(\sigma S) \circ_{\sigma(i)} (\pi F)$ is the $(\sigma(j-m+1)+m-1)$th boundary component of $S \circ_i F$.

Thus $(\sigma S) \circ_{\sigma(i)} (\pi F) = (\sigma \circ_i \pi)(S \circ_i F)$ and the first axiom of an operad is verified.

For the second, let $(S, \rho) \in \mathcal{S}_k$, $(F, \rho') \in \mathcal{S}_l$ and $(K, \rho'') \in \mathcal{S}_m$. Suppose that i and j are such that $1 \leq i \leq j < k$. Consider the $(j + l - 1)$th boundary component of the surface $S \circ_i F$. This is past the boundary components of F, and can be seen to be the jth boundary component of S. Hence if we glue the 0th boundary component of K to this boundary component, we are essentially gluing the root of K to the jth boundary component of S. The surface F is still attached to the ith boundary component of the resulting surface, however.

Hence the second axiom is verified. The third is entirely similar to the second, so we now move on to show that not only is $\{S_n\}_{n \geq 0}$ an operad, it is also a cyclic operad.

The Σ_{n+} action on $\mathcal{S}_n$ is given by permuting all labels. Let $(S, \rho) \in \mathcal{S}_n$ and $(F, \rho') \in \mathcal{S}_m$. Recall that $\tau = (01...n) \in \Sigma_{n+}$. The root of τS is the nth boundary component of S, and the first boundary component of τF is the root of F. Hence $(\tau F) \circ_1 (\tau S)$ is equal to $\tau(S \circ_n b)$ and $\{\mathcal{S}_n\}_{n \geq 0}$ is indeed a cyclic operad.

Finally, note that $\{\mathcal{S}\}_{n \geq 0}$ is by construction graded and stable. $\square$

Recall from Theorem 2.4.4 that a modular operad is a graded operad with contraction maps $\xi_{ij} : \mathcal{S}_{g,n} \to \mathcal{S}_{g+1,n-2}$. These contraction maps are defined similarly to the operadic composition. Specifically, $\xi_{ij}(S)$ is the surface $S/\sim_\beta$ described in the previous section.

Theorem 5.2.2. *Let Σ_n act on $\mathcal{S}_{g,n}$ by permuting labels. Then this action along with the composition and contraction maps make $\mathcal{S}$ into a modular operad in **Grpd**.*

Proof. By Theorem 2.4.4 and Lemma 5.2.1 we simply need to show that the contraction maps satisfy the relations given in Theorem 2.4.4. That is, we need to show the following:

- The $\xi_{i,j}$ are equivariant
- For any finite set I and distinct elements $i, j, k, l \in I$ we have $\xi_{i,j} \circ \xi_{k,l} = \xi_{k,l} \circ \xi_{i,j}$
- If $(S, \rho) \in \mathcal{S}_n$ and $(F, \rho') \in \mathcal{S}_m$ then $\xi_{1,2}(S \circ_n F) = (\xi_{1,2}S) \circ_n F$
- With S, F as before, we have $\xi_{n,n+1}(S \circ_n F) = S \circ_n (\xi_{1,2}F)$
- With S, F as before, we have $\xi_{n-1,n}(S \circ_n F) = \xi_{n+m-2,n+m-1}(S \circ_{n-1} \tau^{-1}F)$ where $\tau = (01...n) \in \Sigma_{n+}$.

To verify that the contraction maps satisfy the relations from Theorem 2.4.4 first note that $\xi_{i,j}$ is clearly equivariant. The second relation is similarly clear.

For the rest, let $(S, \rho) \in \mathcal{S}_n$ and $(F, \rho') \in \mathcal{S}_m$. The first and second boundary components of $S \circ_n F$ are just the first and second boundary components of S, and so $\xi_{1,2}(S \circ_n F) = (\xi_{1,2}S) \circ_n F$. The n and $(n + 1)$th boundary components of $S \circ_n F$ are the first and second boundary components of F, and so $\xi_{n,n+1}(S \circ_n F) = S \circ_n (\xi_{1,2}F)$.

Finally, consider $\xi_{n-1,n}(S \circ_n F)$. Here we are gluing the $(n-1)$th boundary component of S to the first boundary component of F in $S \circ_n F$. Note that $S \circ_{n-1} \tau^{-1}F$ is the result of gluing the $(n - 1)$th boundary component of S to the root of $\tau^{-1}F$, which is just the first boundary component of F. The $(n + m - 2)$th boundary component of $S \circ_{n-1} \tau^{-1}F$ is the mth boundary component of $\tau^{-1}F$ which is just the root of F. Similarly the $(n+m-1)$th boundary component of $S \circ_{n-1} \tau^{-1}F$ is the nth boundary component of S. Hence we can see that $\xi_{n-1,n}(S \circ_n F) = \xi_{n+m-2,n+m-1}(S \circ_{n-1} \tau^{-1}F)$ as required.

Thus we have shown that $\{\mathcal{S}_n\}_{n \geq 0}$ is a stable graded cyclic operad with contraction maps that satisfy the relations from Theorem 2.4.4. Hence it is in fact a modular operad, as desired. □

Definition 5.2.3. The functor $\Gamma : \mathcal{U}_{st}^{op} \to \mathbf{Grpd}$ is given by applying the construction in Theorem 3.5.3 to the modular operad in Theorem 5.2.2.

Expanding on this definition, we have that $\Gamma(G) \cong \otimes_{v \in V} \mathcal{S}_{g(v),|v|}$. If $f : G_0 \to G_1$ is given by substituting a barbell, $\Gamma(f)$ is the corresponding composition map $\circ_{ij}$. Similarly if $f : G_0 \to G_1$ is given by substituting a contracted star, $\Gamma(f)$ is given by the contraction maps $\xi_{i,j}$.

Corollary 5.2.4. The functor Γ restricted to the subcategory of $\mathcal{U}_{st}$ whose objects are all trees gives a cyclic operad, which we denote by Γ_0. This is called the genus 0 surface operad.

Proof. This is essentially just Corollary 2.4.3. □

Chapter 6

Profinite completion and an ∞-cyclic operad

In this section we compose the functor Γ with the classifying space and profinite completion functors to obtain a functor $\widehat{B\Gamma} : \mathcal{U}_{st}^{op} \to \widehat{\mathbf{sSet}}$. Profinite completion will cause the resulting functor to fail to be modular. We will show, however, that $\widehat{B\Gamma_0}$ is an ∞-cyclic operad.

6.1 Goodness of mapping class groups

In this section we outline the current progress on proving that mapping class groups of surfaces are good. Specifically, we shall provide a proof that the mapping class groups are good for genus less than 3.

Definition 6.1.1. A group G is called *fully residually free* if for all finite subsets $X \subseteq G$ such that $1 \notin X$ there is an epimorphism $\varphi : G \to F$, where F is free, such that $1 \notin \varphi(X)$. Equivalently, there exists a normal subgroup $N \triangleleft G$ such that G/N is free and $X \cap N = \emptyset$.

A *limit group* is a finitely generated fully residually free group.

We need the following two theorems about limit groups:

Theorem 6.1.2. *Limit groups are good.*

Proof. See [GJ-ZZ08], Theorem 1.3. $\square$

Theorem 6.1.3. *Let F be a finitely generated free group and $\varphi : F \to F$ an automorphism. Then if $u \in F$ is not a proper power, the free product with amalgamation $F *_{u=\varphi(u)} F$ is a limit group.*

Proof. See [CG05], Corollary 3.6. $\square$

Limit groups also behave well with respect to taking subgroups

Lemma 6.1.4. Finitely generated subgroups of limit groups are limit groups.

Proof. Let G be a limit group and H a subgroup. Let $X \subseteq H$ be any finite subset and let $N \triangleleft G$ be a normal subgroup such that $N \cap X = \emptyset$. Then $N \cap H$ is normal in H and as G/N is free and $H/N \cap H \leq G/N$, we have that $H/N \cap H$ is free. Thus, since H is finitely generated by assumption, H is a limit group. $\square$

Lemma 6.1.5. $\pi_1(S_{g,n})$ are limit groups for all $g, n \geq 0$, where $S_{g,n}$ is the orientable surface of genus g and n boundary components. In particular, they are good.

Proof. As fundamental groups of surfaces are finitely generated it suffices to show that they are fully residually free by Theorem 6.1.2. Note that for $n > 0$ this is trivial as $\pi_1(S_{g,n}) \cong F_{2g+n-1}$ since $S_{g,n} \simeq \bigvee^{2g+n-1} S^1$.

Thus we only need to prove the result for closed surfaces S_g. For $g = 0, 1$ the result is again trivial. For $g > 2$, note that as S_g is a $(g-1)$-fold covering space of S_2, $\pi_1(S_g)$ is a subgroup of $\pi_1(S_2)$. Thus by Lemma 6.1.4 it suffices to prove the result for $g = 2$.

To see this, simply note that $\pi_1(S_2) \cong \langle a_1, b_1 \rangle *_{[a_1,b_1]=[b_2,a_2]} \langle b_2, a_2 \rangle$. Thus by Theorem 6.1.3, $\pi_1(S_2)$ is a limit group. $\square$

Recall the Birman exact sequence:

$$1 \longrightarrow \pi_1(S_{g,n}) \xrightarrow{Push} \Gamma^1_{g,n} \xrightarrow{\alpha} \Gamma_{g,n} \longrightarrow 1$$

This allows us to prove that goodness of the general mapping class group is determined by goodness of the mapping class groups of closed surfaces.

Theorem 6.1.6. *If $\Gamma_{g,0}$ is good then $\Gamma_{g,n}$ is good for all $n \geq 0$.*

Proof. This is a simple case of induction using the Birman exact sequence and Theorem 4.3.4. Specifically, by Lemma 6.1.5 the fundamental groups of surfaces are good and by Lemma 4.3.5 the cohomology groups $H^q(\pi_1(S_{g,n}), M)$ are finite if M is finite. Hence Theorem 4.3.4 applied to the Birman exact sequence gives that $\Gamma^1_{g,n}$ is good.

Next, note that there is an exact sequence

$$1 \longrightarrow \langle T_\gamma \rangle \longrightarrow \Gamma_{g,n+1} \longrightarrow \Gamma^1_{g,n} \longrightarrow 1$$

where T_γ is a Dehn twist around a boundary component of $S_{g,n+1}$. Since $\Gamma_{g,n+1}$ is torsion free, $\langle T_\gamma \rangle \cong \mathbb{Z}$. Since S^1, the classifying space of $\mathbb{Z}$, is finite, applying Theorem 4.3.4 and Lemma 4.3.5 once more gives that $\Gamma_{g,n+1}$ is good. □

Hence to show goodness of mapping class groups we only need to consider the case of a closed surface. As an application of this principle, we can show that mapping class groups of surfaces with genus less than 3 are good.

Corollary 6.1.7. The mapping class groups $\Gamma_{g,n}$ are good if $g \leq 2$.

Proof. First, note that $\mathbb{Z}/2\mathbb{Z}$ is good because $\widehat{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$. Also, the classifying space of $\mathbb{Z}/2\mathbb{Z}$ is $\mathbb{R}P^\infty$, which has exactly one cell in every dimension. Hence by Lemma 4.3.5 the cohomology groups $H^q(\mathbb{Z}/2\mathbb{Z}, M)$ are finite for any finite $\mathbb{Z}/2\mathbb{Z}$-module M.

Now, $\Gamma_{0,0}$ is the trivial group and is clearly good. Hence $\Gamma_{0,n}$ is good for all n by Theorem 6.1.6. Furthermore, by Theorem 1.1 in [GJ-ZZ08], $PSL_2(\mathbb{Z})$ is good. This implies that $SL_2(\mathbb{Z})$ is good as $SL_2(\mathbb{Z})$ is a central extension of $PSL_2(\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$. Hence $\Gamma_{1,0} \cong SL_2(\mathbb{Z})$ is good and so is $\Gamma_{1,n}$ for all n.

Finally, theorem 5 in [BH73] shows that there is an exact sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \Gamma_2^0 \longrightarrow \Gamma_0^6 \longrightarrow 1$$

Hence we have that Γ_2^0 is good as Γ_0^6 is. Since $\Gamma_{2,0} \cong \Gamma_2^0$, we have that $\Gamma_{2,0}$ is good and hence that $\Gamma_{2,n}$ is good for all $n \geq 0$. □

6.2 The profinite completion of $B\Gamma$

Recall from the introduction that we would like the profinite completion of $B\Gamma$ to be a model for the Teichmüller tower. Specifically, we want the profinite completion of $B\Gamma$ to be an ∞-modular operad.

The main obstruction to proving this is that we do not know whether the mapping class groups for $g > 2$ are good. There are several approaches one could take; invoking more general theory or playing with adjoint functors as in [Qui08] to show that the profinite completion has the automorphisms of an ∞-modular operad, which suffices for the intended application to Grothendieck's *esquisse d'un programme*.

Instead, we opt to prove a weaker result using the more elementary methods available in this thesis. We know that the genus 0 mapping class groups are good, which allows us to prove that the profinite completion of the genus 0 surface operad (defined in Corollary 5.2.4) is an ∞-cyclic operad.

Theorem 6.2.1. *$\widehat{B\Gamma_0}$ is an ∞-cyclic operad.*

Proof. This is almost entirely similar to the proof of Theorem 4.4.2. First, recall from Lemma 4.3.3 that finite products of good groups are good. Hence

$\Gamma_T \cong \prod_{v\in V} \Gamma_{*_v}$ is good for all trees T. Thus $\widehat{B\Gamma_T}$ is fibrant for all trees T as it is the classifying space of $\widehat{\prod_{v\in V} \Gamma_{*_v}}$. Also, Γ_I is the trivial group and so $\widehat{B\Gamma_I}$ is contractible.

Finally, we also have

$$\widehat{B\Gamma_T} \cong \widehat{\prod_{v\in V} B\Gamma_{*_v}} \simeq \prod_{v\in V} \widehat{B\Gamma_{*_v}}$$

by Theorem 4.3.2. Thus $\widehat{B\Gamma_0}$ is an ∞-cyclic operad. □

Remark. It is worth noting that the proof of Theorem 6.2.1 will work exactly the same for higher genus if it were proven that $\Gamma_{g,n}$ is good for all g and n. Hence if it were proven that $\Gamma_{g,0}$ is good for all $g \geq 0$, we would have that $\widehat{B\Gamma}$ is an ∞-modular operad, as desired for its applications to the Teichmüller tower.

Bibliography

[Bel80] G. Belyi, *On Galois extensions of a maximal cyclotomic field*, Math. USSR Izv. 14 (1980) pp. 247–256

[BH73] J. Birman, H. Hilden, *On isotopies of homeomorphisms of Riemann surfaces.* Ann. of Math. (2) 97 (1973), 424–439.

[BdBHR18] P. B. de Brito, G. Horel, M. Robertson, *Operads of genus 0 curves and the Grothendieck-Teichmüller group.* To appear in Geometry & Topology.

[CG05] C. Champetier, V. Guirardel, *Limit groups as limits of free groups: Compactifying the set of free groups.* Israel Journal of Mathematics, The Hebrew University Magnes Press, 2005, 146, pp.1.

[Dri90] V. Drinfeld, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, Algebra i Analiz. **2** (1990), no. 4, 149–181.

[FM12] B. Farb, D. Margalit, *A primer on mapping class groups.* Princeton Mathematical Series, 49. Princeton University Press, Princeton, NJ, 2012. xiv+472 pp. ISBN: 978-0-691-14794-9

[Get95] E. Getzler, *Operads and moduli spaces of genus* 0 *Riemann surfaces.* The moduli space of curves (Texel Island, 1994), 199-230, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.

[GK98] E. Getzler, M.M. Kapranov, *Modular Operads*. Compositio Math. vol. 110, 1998, no. 1, 65–126.

[Gro97] A. Grothendieck, *Esquisse d'un programme.* With an English translation on pp. 243-283. London Math. Soc. Lecture Note Ser., 242, Geometric Galois actions, 1, 548, Cambridge Univ. Press, Cambridge, 1997.

[GJ-ZZ08] F. Grunewald, A. Jaikin-Zapirain, P.A. Zalesskii, *Cohomological goodness and the profinite completion of Bianchi groups*. Cohomological goodness and the profinite completion of Bianchi groups. Duke Math. J. 144 (2008), no. 1, 53–72.

[HRY18] P. Hackney, M.Robertson and D. Yau, *A Segal model for modular operads and compact symmetric multicategories*. Paper in Preparation.

[JK11] A. Joyal and J. Kock, *Feynman graphs, and nerve theorem for compact symmetric multicategories (Extended Abstract)*, Electronic Notes in Theoretical Computer Science, vol. 270, no. 2, 105–113 (2011).

[Qui08] G. Quick, *Profinite homotopy theory*, Documenta Mathematica, 13 (2008), 585–612.

[Qui12] G. Quick, *Some remarks on the profinite completion of spaces*, Galois-Teichmüller theory and Arithmetic Geometry, Advanced Studies in Pure Mathematics, vol. 63, Mathematical Society of Japan (2012), pp. 413-448 .

[Seg68] G. Segal, *Classifying spaces and spectral sequences*. Inst. Hautes Études Sci. Publ. Math. No. 34 1968 105–112.

[Sul74] D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*. Annals of Mathematics **100** (1974), no. 1, pp. 1–79.

[Til00] U. Tillmann, *Higher genus surface operad detects infinite loop spaces*. Mathematische Annalen, **317** (2000), no. 3, pp. 613–628.

[Wei94] C. Weibel, *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994. xiv+450 pp. ISBN: 0-521-43500-5; 0-521-55987-1

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